Bachelor of Science

(B.SC- PCM)

Differential Equation

(DBSPCO203T24)

Self-Learning Material
(SEM II)

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Course Code: DBSPCO203T24
Differential Equation

TABLE OF CONTENT

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COURSE INTRODUCTION

Differential equation is a foundational branch of mathematics with far-reaching implications in various fields, including physics, engineering, economics, and computer science. It serves as a fundamental tool for understanding rates of change, optimization, and the behaviour of functions. The course is divided into 12 units. Each Unit is divided into sub topics.

The Units provide students with a comprehensive understanding of the differential equation and its types and method of solution. These are just a few examples. The power of differential equations lies in their ability to model dynamic systems and predict future behavior based on initial conditions and parameters..

There are sections and sub-sections inside each unit. Each unit starts with a statement of objectives that outlines the goals we hope you will accomplish. Every segment of the unit has many tasks that you need to complete.

We wish you pleasure in the Course. We are certain that you will get better at math if you follow through on it.

Course Outcomes: After the completion of the course, the students will be able to

1. Recall the concepts of ordinary differential equations (ODEs), partial differential equations (PDEs), initial and boundary conditions..
2. Explain Differentiate between linear and nonlinear equations, and recognize homogeneous and nonhomogeneous equations.
3. Apply and solve first-order ODEs using methods such as separation of variables, integrating factors, and exact equations..
4. Analyze higher-order linear ODEs and apply methods like reduction of order and solving with special functions.
5. Evaluate general solutions to second-order linear homogeneous equations with constant coefficients and nonhomogeneous terms using methods such as undetermined coefficients and variation of parameters.
6. Create methods to solve PDEs and analyze periodic functions.

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Unit-1

Introduction to Differential Equations

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure:

1.1 Differential Equations:

Differential equations are those that have a differential coefficient..

For Example,

1. $\frac{dy}{dx}=\frac{1+x^2}{1-y^2}$ 2. $\frac{d^2y}{dx^2}=2\frac{dy}{dx}-8y=0$ 3. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=k\frac{d^2y}{dx^2}$

4. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu,$ 5. $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial z}{\partial y}$

1.2 Order and degree of a differential equation:-

A differential equation's order is the highest differential coefficient that is present in it, and its degree is the highest derivative that remains after the radical sign and fraction have been eliminated.

1. $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E\sin wt.$ 2. $\cos x\frac{d^2y}{dx^2}+\sin x\left(\frac{dy}{dx}\right)^2+8y=\tan x$

3. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$

Order is 2.

Degree of the Eqn (1) and (2) is 1.

Degree of the Eqn(3) is 2.

1.3 Formation of Differential Equation:-

The ordinary equation can be differentially expressed and the arbitrary constants removed to form the differential equations.

Ex. 1: Eliminates the arbitrary constants and find the order.

(a) $y = Ax + A^2$ (b) $y = A\cos x + B\sin x$ (c) $y^2 = Ax^2 + Bx + C.$

Solution. (*a*) $y = Ax + A^2$ ... (1)

On differentiation $\frac{dy}{dx} = A$

Putting the value of A in (1), we get $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$ **Ans.**

On eliminating one constant A we get the differential equation of order 1.

(*b*) $y = A \cos x + B \sin x$

On differentiation $\frac{dy}{dx} = -A\sin x + B\cos x$

Again differentiating

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x \Rightarrow \frac{d^2y}{dx^2} = -(A\cos x + B\sin x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0 \quad \textbf{Ans.}$$

This is differential equation of order 2 obtained by eliminating two constants A and B.

(*c*) $y^2 = Ax^2 + Bx + C$

On differentiation $2y\frac{dy}{dx} = 2Ax + B$

Again differentiating $2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2A$

On differentiating again $y\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 0 \Rightarrow y\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\frac{d^2y}{dx^2} = 0$ **Ans.**

This is the differential equation of order 3, obtained by eliminating three constants A, B, C.

Exercise

1. Write the order and the degree of the following differential equations.

(*i*) $\frac{d^2y}{dx^2} + a^2x = 0;$ (*ii*) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2};$ (*iii*) $x^2\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^4 = 0.$

Ans. (*i*) 2,1 (*ii*) 2,2 (*iii*) 2,3

2. Give an example of each of the following type of differential equations.

(*i*) A linear-differential equation of second order and first degree **Ans.** *Q*, 1 (*i*)

(*ii*) A non-linear differential equation of second order and second degree **Ans.** *Q*, 1 (*ii*)

(*iii*) Second order and third degree. **Ans.** *Q* 1 (*iii*)

3. Obtain the differential equation of which $y^2 = 4a(x + a)$ is a solution.

Ans. $y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} - y^2 = 0$

4. Obtain the differential equation associated with the primitive $Ax^2 + By^2 = 1$.

Ans. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

5. Find the differential equation corresponding to

$y = a\,e^{3x} + be^x$.

Ans. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

6. By the elimination of constants A and B, find the differential equation of which

$y = e^x\,(A\cos x + B\sin x)$ is a solution.

Ans. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

7. Find the differential equation whose solution is $y = a\cos(x = 3)$. (*A.M.I.E., Summer 2000*)

Ans. $\frac{dy}{dx} = -\tan(x+3)$

8. Show that set of function $\left\{x, \frac{1}{x}\right\}$ forms a basis of the differential equation $x^2y'' + xy' - y = 0$.

Obtain a particular solution when $y(1) = 1$, $y(1) = 2$.

Ans. $y = \frac{3x}{2} - \frac{1}{2x}$

1.4 Solution of a Differential Equation:-

In the example 1(*b*), $y = A\cos x + B\sin x$, on eliminating A and B we get the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

$y = A\cos x + B\sin x$ is called the solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

1.5 Differential Equations of the first order and first degree:-

The standard methods of solving the differential equations are

(i) Separation of the variables.

(ii) Homogeneous equations.

(iii) Linear equations of the first order.

(iv) Exact differential equations.

Variable Problem

If a differential equation can be written in the form

$$f(y)\,dy = \phi(x)\,dx$$

We say that variables are separable, y on left hand side and x on right hand side. We get the solution by integrating both sides.

Working Rule:

Step 1. Separate the variables as $f(y)\,dy = \phi(x)\,dx$

Step 2. Integrate both sides as $\int f(y)\,dy = \int \phi(x)\,dx$

Step 3. Add an arbitrary constant C on R.H.S.

Example 2. *Solve* : $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$

Solution. We have, $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$

Separating the variables, we get

$$(\sin y + y\cos y)\,dy = \{x\,(2\log x + 1)\}\,dx$$

Integrating both the sides, we get $\int (\sin y + y\cos y)dy = \int \{x(2\log x + 1)\}\,dx + C$

$$-\cos y + y\sin y - \int (1)\cdot \sin y\,dy = 2\int \log x \cdot x\,dx + \int x\,dx + C$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y = 2\left[\log x \cdot \frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}dx\right] + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x \cdot \frac{x^2}{2} - \int x\,dx + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = 2\log x \cdot \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = x^2\log x + C$$

Ans.

Example 3. *Solve the differential equation.*

$$x^4\frac{dy}{dx}+x^3y=-\sec(x\,y).$$

Solution. $x^4\frac{dy}{dx}+x^3y=-\sec(x\,y) \quad \Rightarrow \quad x^3\left(x\frac{dy}{dx}+y\right)=-\sec xy$

Put $\quad v=xy,\ \frac{dv}{dx}=x\frac{dy}{dx}+y \quad \Rightarrow \quad x^3\frac{dv}{dx}=-\sec v$

$\Rightarrow \quad \frac{dv}{\sec v}=-\frac{dx}{x^3} \qquad \Rightarrow \int\cos v\,dv=-\int\frac{dx}{x^3}+c$

$\Rightarrow \ \sin v=\frac{1}{2x^2}+c \qquad \Rightarrow \ \sin xy=\frac{1}{2x^2}+c$ **Ans.**

Example 4. *Solve* : $\quad \cos(x+y)dy=dx$

Solution. $\quad \cos(x+y)\,dy=dx \quad \Rightarrow \quad \frac{dy}{dx}=\sec(x+y)$

On putting $\quad x+y=z$

So that $\quad 1+\frac{dy}{dx}=\frac{dz}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{dz}{dx}-1$

$$\frac{dz}{dx}-1=\sec z \quad \Rightarrow \quad \frac{dz}{dx}=1+\sec z$$

Separating the variables, we get

$$\frac{dz}{1+\sec z}=dx$$

On integrating,

$$\int\frac{\cos z}{\cos z+1}dz=\int dx \quad \Rightarrow \quad \int\left[1-\frac{1}{\cos z+1}\right]dz=x+C$$

$$\int\left(1-\frac{1}{2\cos^2\frac{z}{2}-1+1}\right)dz=x+C$$

$$\int\left(1-\frac{1}{2}\sec^2\frac{z}{2}\right)dz=x+C \quad \Rightarrow \quad z-\tan\frac{z}{2}=x+C$$

$$x + y - \tan\frac{x+y}{2} = x + C$$

$$y - \tan\frac{x+y}{2} = C$$ **Ans.**

Example 5. *Solve the equation.*

$(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$ (*U.P. II Semester, Summer 2005*)

Solution. We have

$$(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$$

Re-arranging (1), we get $\dfrac{x\,dx}{y\,dy} = \dfrac{3x^2 + 2y^2 - 8}{2x^2 + 3y^2 - 7}$

Applying componendo and dividendo rule, we get

$$\frac{x\,dx + y\,dy}{x\,dx - y\,dy} = \frac{5x^2 + 5y^2 - 15}{x^2 - y^2 - 1} \quad \Rightarrow \quad \frac{x\,dx + y\,dy}{x^2 + y^2 - 3} = 5\left(\frac{x\,dx - y\,dy}{x^2 - y^2 - 1}\right)$$

Multiplying by 2 both the sides, we get

$$\Rightarrow \quad \left(\frac{2x\,dx + 2y\,dy}{x^2 + y^2 - 3}\right) = 5\left(\frac{2x\,dx - 2y\,dy}{x^2 - y^2 - 1}\right)$$

Integrating both sides, we get

$$\log(x^2 + y^2 - 3) = 5\log(x^2 - y^2 - 1) + \log C$$

$$\Rightarrow \quad x^2 + y^2 - 3 = C\,(x^2 - y^2 - 1)^5$$ **Ans.**

ıere C is arbitrary constant of integration.

Solve the following differential equations :

1. $\dfrac{dx}{x} = \tan y \cdot dy$ **Ans.** $x \cos y = C$
2. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ **Ans.** $\sin^{-1} y = \sin^{-1} x + C$
3. $y(1+x^2)^{1/2}dy + x\sqrt{1+y^2}\,dx = 0$ **Ans.** $\sqrt{1+y^2} + \sqrt{1+x^2} = C$
4. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$ **Ans.** $\tan x \tan y = C$
5. $(1 + x^2)\,dy - x\,y\,dx = 0$ **Ans.** $y^2 = C\,(1 + x^2)$
6. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$ **Ans.** $(e^y + 1)\sin x = C$
7. $3\,e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$ **Ans.** $(1 - e^x)^3 = C \tan y$
8. $(e^y + 2)\sin x\,dx - e^y \cos x\,dy = 0$ **Ans.** $(e^y + 2)\cos x = C$
9. $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ **Ans.** $e^y = e^x + \dfrac{x^3}{3} + C$
10. $\dfrac{dy}{dx} = 1 + \tan(y - x)$ [Put $y - x = z$] **Ans.** $\sin(y - x) = e^{x+c}$
11. $(4x + y)^2 \dfrac{dx}{dy} = 1$ **Ans.** $\tan^{-1}\dfrac{4x+y}{2} = 2x + C$
12. $\dfrac{dy}{dx} = (4x + y + 1)^2$ [**Hint.** Put $4x + y + 1 = z$] **Ans.** $\tan^{-1}\dfrac{4x+y+1}{2} = 2x + C$

HOMOGENEOUS DIFFENTIAL EQUATIONS :

A diffential equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$$

Example 6. *Solve the following differential equation*

$$(2xy + x^2)\, y = 3y^2 + 2xy$$

Solution. We have, $(2xy + x^2)\dfrac{dy}{dx} = 3y^2 + 2xy \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{3y^2 + 2xy}{2xy + x^2}$

Put $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

On substituting, the given equation becomes $v + x\dfrac{dv}{dx} = \dfrac{3v^2x^2 + 2vx^2}{2vx^2 + x^2} = \dfrac{3v^2 + 2v}{2v+1}$

$\Rightarrow \quad x\dfrac{dv}{dx} = \dfrac{3v^2 + 2v - 2v^2 - v}{2v+1} \qquad \Rightarrow \quad x\dfrac{dv}{dx} = \dfrac{v^2 + v}{2v+1} \;\Rightarrow\; \left(\dfrac{2v+1}{v^2+v}\right)dv = \dfrac{dx}{x}$

$\Rightarrow \quad \displaystyle\int\left(\frac{2v+1}{v^2+v}\right)dv = \int\frac{dx}{x} \qquad \Rightarrow \quad \log(v^2 + v)\ \log x + \log c$

$\Rightarrow \quad v^2 + v = cx \qquad \Rightarrow \quad \dfrac{y^2}{x^2} + \dfrac{y}{x} = cx$

$\Rightarrow \quad y^2 + xy = cx^3$

Example 7. *Solve the equation* :

$$\frac{dy}{dx} = \frac{y}{x} + x\sin\frac{y}{x}$$

Solution. $\qquad \dfrac{dy}{dx} = \dfrac{y}{x} + x\sin\dfrac{y}{x}$ ... (1)

Put $\qquad y = vx$ in (1) so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = v + x\sin v$$

$\Rightarrow \qquad x\dfrac{dv}{dx} = x\sin v \qquad \Rightarrow \qquad \dfrac{dv}{dx} = \sin v$

Separating the variable, we get

$\Rightarrow \qquad \dfrac{dv}{\sin v} = dx \qquad \Rightarrow \qquad \displaystyle\int \operatorname{cosec} v\, dv = \int dx + C$

$\log\tan\dfrac{v}{2} = x + C \qquad \Rightarrow \qquad \log\tan\dfrac{y}{2x} = x + C$ **Ans.**

Exercise

Solve the following

1. $(y^2 - xy)\,dx + x^2 dy = 0$ **Ans.** $\frac{x}{y} = \log x + C$
2. $(x^2 - y^2)\,dx + 2xy\,dy = 0$ **Ans.** $x^2 + y^2 = ax$
3. $x(y-x)\frac{dy}{dx} = y(y+x).$ **Ans.** $\frac{y}{x} - \log xy = a$
4. $x(x-y)\,dy + y^2\,dx = 0$ **Ans.** $y = x \log C y$

5. $\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$ **Ans.** $y - x = C(x+y)^3$
6. $\frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x}$ **Ans.** $\sin\frac{y}{x} = Cx$
7. $\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$ **Ans.** $3x + y\log x + Cy = 0$
8. $\frac{dy}{dx} = \frac{x^2 - 2y^2}{2xy}$ **Ans.** $4y^2 - x^2 = \frac{C}{x^2}$
9. $(x^2 + y^2)\,dy = xy\,dx$ **Ans.** $-\frac{x^2}{2y^2} + \log y = C$
10. $x^2 y\,dx - (x^3 + y^3)\,dy = 0$ **Ans.** $\frac{-x^3}{3y^3} + \log y = C$

EQUATION REDUCIBLE TO HOMOGENEOUS FORM :

Example 8. *Solve* : $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Solution. Put $x = X + h, \quad y = Y + k.$

The given equation reduces to

$$\therefore \quad \frac{dY}{dX} = \frac{(X+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3} \qquad \left(\frac{1}{2} \neq \frac{2}{1}\right)$$

$$= \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \qquad \text{.... (1)}$$

Now choose h and k so that $h + 2k - 3 = 0,\ 2h + k - 3 = 0$

Solving these equations we get $h = k = 1$

$$\therefore \quad \frac{dY}{dX} = \frac{X+2Y}{2X+Y} \qquad \text{... (2)}$$

Put $Y = vX$, so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$

The equation (2) is transformed as

$$v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX} = \frac{1+2v}{2+v}$$

$$X\frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1-v^2}{2+v} \quad \Rightarrow \quad \left(\frac{2+v}{1-v^2}\right)dv = \frac{dX}{X}$$

$$\Rightarrow \quad \frac{1}{2}\frac{1}{(1+v)}dv + \frac{3}{2}\frac{1}{1-v}dv = \frac{dX}{X} \qquad \text{(Partial fractions)}$$

On integrating, we have

$$\frac{1}{2}\log(1+v) - \frac{3}{2}\log(1-v) = \log X + \log C$$

$$\Rightarrow \quad \log \frac{1+v}{(1-v)^3} = \log C^2 X^2 \quad \Rightarrow \quad \frac{1+v}{(1-v)^3} = C^2 X^2$$

$$\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3} = C^2 X^2 \quad \Rightarrow \quad \frac{X+Y}{(X-Y)^3} = C^2 \text{ or } X + Y = C^2 (X - Y)^3$$

Put $X = x - 1$ and $Y = y - 1$ $\Rightarrow$ $x + y - 2 = a (x - y)^3$ **Ans.**

Example 9. *Solve* : $(x + 2y) (dx - dy) = dx + dy$

Solution. $(x + 2y) (dx - dy) = dx + dy$ $\Rightarrow$ $(x + 2y - 1) dx - (x + 2y + 1) dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+2y-1}{x+2y+1} \quad \text{...(1)}$$

Hence $\frac{a}{A} = \frac{b}{B}$ *i.e.*, $\left(\frac{1}{1} = \frac{2}{2}\right)$ (Case of failure)

Now put $x + 2y = z$ so that $1 + 2\frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) becomes

$$\frac{1}{2}\frac{dz}{dx} - \frac{1}{2} = \frac{z-1}{z+1} \quad \Rightarrow \quad \frac{dz}{dx} = 2\frac{(z-1)}{z+1} + 1 = \frac{3z-1}{z+1}$$

$$\Rightarrow \quad \frac{z+1}{3z-1} dz = dx \quad \Rightarrow \quad \left(\frac{1}{3} + \frac{4}{3}\frac{1}{3z-1}\right) dz = dx$$

On integrating, $\frac{z}{3} + \frac{4}{9}\log(3z-1) = x + C$

$$3z + 4 \log (3z - 1) = 9x + 9C$$

$$\Rightarrow \quad 3 (x + 2y) + 4 \log (3x + 6y - 1) = 9x + 9C$$

$$3x - 3y + a = 2 \log (3x + 6y - 1)$$ **Ans.**

1.6 Summary

Differential equations are mathematical formulas that use derivatives to explain the relationship between a function and its derivatives. Ordinary differential equations (ODEs) are used to solve functions with a single variable. ODEs are created by expressing the derivatives of an unknown function in terms of the independent variable and the function itself. ODEs of first order and first degree involve the first derivative of an unknown function. Variable separable is a method for solving ODEs in which the variables may be separated on either side of the problem. It entails decoupling the variables, integrating both sides, and arriving at a general solution.

1.7Keywords

- Ordinary differential equations

- Variable separable
- Homogeneous

1.8Self-Assessment Questions

Solve:

1. $\frac{dy}{dx} = \frac{2x+9y-20}{6x+2y-10}$ **Ans.** $(2x - y)^2 = C\,(x + 2y - 5)$
2. $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$ **Ans.** $\log[(y+3)^2+(x+2)^2]+2\tan^{-1}\frac{y+3}{x+2}=a$
3. $\frac{dy}{dx} = \frac{x-y-2}{x+y+6}$ **Ans.** $(y + 4)^2 + 2\,(x + 2)\,(y + 4) - (x + 2)^2 = a^2$
4. $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ **Ans.** $-\,(y - 3)^2 + 2(x + 1)\,(y - 3) + (x + 1)^2 = a$
5. $\frac{dy}{dx} = \frac{2x-5y+3}{2x+4y-6}$ **Ans.** $(x - 4y + 3)\,(2x + y - 3) = a$
6. $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$ **Ans.** $2\,(2x + y) + \log\,(2x + y - 1) = 3x + C$
7. $(x - y - 2)\,dx - (2x - 2y - 3)\,dy = 0$ **Ans.** $\log\,(x - y - 1) = x - 2y + C$

1.9Case Study

A is a biologist who is researching the population dynamics of a certain species in an ecosystem. The species' population is influenced by a variety of factors, including birth rate, mortality rate, and accessible resources. Your objective is to use differential equations to simulate population increase and analyze population behavior over time.

Question

1. Given the birth rate of the species as 0.05 individuals per day and the mortality rate as 0.03 individuals per day, calculate the net population growth rate per day.
2. Suppose the accessible resources for the species decrease over time, causing a decline in the birth rate from 0.05 to 0.03 individuals per day. If the mortality rate remains constant at 0.03 individuals per day, calculate the new net population growth rate and the equilibrium population size assuming the birth and mortality rates remain constant.

1.10 References

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2 Gupta, S. P and Kapoor V.K, Fundamental of Mathematical Statistics, Sultan Chand and Sons, New Delhi.

Unit-2

Linear Differential Equations

Learning Objectives:

- To understand linear Differential equations
- To understand Equations of first order and first degree
- To understand Bernoulli equation

Structure:

2.1 Linear Differential Equations:-

A differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Working Rule

Step 1. Convert the given equation to the standard form of linear differential equation

i.e. $$\frac{dy}{dx} + Py = Q$$

Step 2. Find the integrating factor i.e. I.F. = $e^{\int P dx}$

Step 3. Then the solution is $y(I.F.) = \int Q\ (I.F.) dx + C$

Ex.1. Evaluate

$$(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$$

Solution. $$\frac{dy}{dx} - \frac{y}{x+1} = e^x (x+1)$$

$$\text{I.F.} = e^{-\int \frac{dx}{x+1}} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

$$y.\frac{1}{x+1} = \int e^x.(x+1).\frac{1}{x+1} dx = \int e^x dx$$

$$\frac{y}{x+1} = e^x + C$$ **Ans.**

Ex 2.Evaluate

$$(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x.$$

Solution. We have $(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{3x^2 - 1}{x^3 - x}y = \frac{x^5 - 2x^3 + x}{x^3 - x} \quad \Rightarrow \quad \frac{dy}{dx} - \frac{3x^2 - 1}{x^3 - x}y = x^2 - 1$$

$$\text{I.F.} = e^{\int -\frac{3x^2 - 1}{x^3 - x}dx} = e^{-\log(x^3 - x)} = e^{\log(x^3 - x)^{-1}} = \frac{1}{x^3 - x}$$

Its solution is

$$y(\text{I.F.}) = \int Q(I.F.)\, dx + C \quad \Rightarrow \quad y\left(\frac{1}{x^3 - x}\right) = \int \frac{x^2 - 1}{x^3 - x} dx + C$$

$$\Rightarrow \quad \frac{y}{x^3 - x} = \int \frac{x^2 - 1}{x(x^2 - 1)} dx + C \quad \Rightarrow \quad \frac{y}{x^3 - x} = \int \frac{1}{x} dx + C$$

$$\Rightarrow \quad \frac{y}{x^3 - x} = \log x + C \quad \Rightarrow \quad y = (x^3 - x)\log x + (x^3 - x)\, C$$ **Ans.**

Ex. 3.Evaluate

$$\sin x\frac{dy}{dx}+2y=\tan^3\left(\frac{x}{2}\right)$$

Solution. Given equation : $\sin x\frac{dy}{dx}+2y=\tan^3\frac{x}{2} \Rightarrow \frac{dy}{dx}+\frac{2}{\sin x}y=\frac{\tan^3\frac{x}{2}}{\sin x}$

This is linear form of $\frac{dy}{dx}+Py=Q$

$\therefore$ $$P=\frac{2}{\sin x} \text{ and } Q=\frac{\tan^3\frac{x}{2}}{\sin x}$$

$\therefore$ $$\text{I.F.}=e^{\int P dx}=e^{\int\frac{2}{\sin x}dx}=e^{2\int\operatorname{cosec}x\,dx}=e^{2\log\tan\frac{x}{2}}=\tan^2\frac{x}{2}$$

$\therefore$ Solution is $$y.(\text{I.F.})=\int \text{I.F.}(Q\,dx)+C$$

$$y\tan^2\frac{x}{2}=\int\tan^2\frac{x}{2}\cdot\frac{\tan^3\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}+C=\frac{1}{2}\int\frac{\tan^4\frac{x}{2}}{\cos^2\frac{x}{2}}dx+C$$

$$=\frac{1}{2}\int\tan^4\frac{x}{2}.\sec^2\frac{x}{2}dx+C \qquad \text{... (1)}$$

Putting $\tan\frac{x}{2}=t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$ on R.H.S. (1), we get

$$y.\tan^2\frac{x}{2}=\frac{1}{2}\int t^4(2dt)+C \Rightarrow y\tan^2\frac{x}{2}=\frac{t^5}{5}+C$$

$$y\tan^2\frac{x}{2}=\frac{\tan^5\frac{x}{2}}{5}+C$$ **Ans.**

Exercise

Solve:

1. $\frac{dy}{dx}+\frac{1}{x}y=x^3-3$ **Ans.** $xy=\frac{x^5}{5}-\frac{3x^2}{2}+C$
2. $(2y-3x)\,dx+x\,dy=0$ **Ans.** $y\,x^2=x^3+C$
3. $\frac{dy}{dx}+y\cot x=\cos x$ **Ans.** $y\sin x=\frac{\sin^2 x}{2}+C$
4. $\frac{dy}{dx}+y\sec x=\tan x$ **Ans.** $y=\frac{C-x}{\sec x+\tan x}+1$
5. $\cos^2 x\frac{dy}{dx}+y=\tan x$ **Ans.** $y=\tan x-1+Ce^{-\tan x}$
6. $(x+a)\frac{dy}{dx}-3y=(x+a)^5$ **Ans.** $2y=(x+a)^5+2C(x+a)^3$
7. $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$ **Ans.** $x\,y=\sin x+C\cos x$
8. $x\log x\frac{dy}{dx}+y=2\log x$ **Ans.** $y\log x=(\log x)^2+C$
9. $x\frac{dy}{dx}+2y=x^2\log x$ **Ans.** $yx^2=\frac{x^4}{4}\log x-\frac{x^4}{16}+C$
10. $dr+(2r\cot\theta+\sin 2\theta)\,d\theta=0$ **Ans.** $r\sin^2\theta=\frac{-\sin^4\theta}{2}+C$

2.2 Bernoulli equation:-

The equation of the form

$$\frac{dy}{dx} + Py = Qy^n \qquad ...(1)$$

where **P** and **Q** are constants or functions of x can be reduced to the linear form on dividing by y^n and substituting $\frac{1}{y^{n-1}} = z$

On dividing bothsides of (1) by y^n, we get

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P = Q \qquad ...(2)$$

Put $\frac{1}{y^{n-1}} = z$, so that $\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{dz}{1-n}$

$\therefore$ (2) becomes $\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$ or $\frac{dz}{dx} + P(1-n)z = Q(1-n)$

Ex4.Evaluate

$x^2 dy + y(x + y)\, dx = 0$

Solution. We have, $x^2\, dy + y\,(x + y)\, dx = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2} \quad \Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -\frac{1}{x^2}$$

Put $-\frac{1}{y} = z$ so that $\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

The given equation reduces to a linear differential equation in z.

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

$$\textbf{I.F.} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log 1/x} = \frac{1}{x}.$$

Hence the solution is

$$z.\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}\, dx + C \quad \Rightarrow \quad \frac{z}{x} = \int -x^{-3}\, dx + C$$

$$\Rightarrow \quad -\frac{1}{xy} = -\frac{x^{-2}}{-2} + C \quad \Rightarrow \quad \frac{1}{xy} = -\frac{1}{2x^2} - C \qquad \textbf{Ans.}$$

Ex 5.Evaluate

$$x\frac{dy}{dx} + y\log y = xy\; e^x$$

Solution. $x\frac{dy}{dx}+y\log y=xy\,e^x$

Dividing by xy, we get

$$\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}\log y=e^x \qquad ...(1)$$

Put $\log y = z$, so that $\frac{1}{y}\frac{dy}{dx}=\frac{dz}{dx}$

Equation (1) becomes, $\frac{dz}{dx}+\frac{z}{x}=e^x$

$$\text{I.F.}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$$

Solution is $zx=\int xe^x\,dx+C$

$$zx=xe^x-e^x+C$$

$\Rightarrow$ $x\log y=xe^x-e^x+C$ **Ans.**

Ex 6.Evaluate

$\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y.$

Solution. $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$

$\Rightarrow$ $\cos y\frac{dy}{dx}-\frac{\sin y}{1+x}=(1+x)e^x \qquad ...(1)$

Put $\sin y = z$, so that $\cos y\frac{dy}{dx}=\frac{dz}{dx}$

(1) becomes $\frac{dz}{dx}-\frac{z}{1+x}=(1+x)e^x$

$$I.F.=e^{-\int\frac{1}{1+x}dx}=e^{-\log(1+x)}=e^{\log\frac{1}{1+x}}=\frac{1}{1+x}$$

Solution is $z.\frac{1}{1+x}=\int(1+x)e^x.\frac{1}{1+x}dx+C=\int e^x dx+C$

$\frac{\sin y}{1+x}=e^x+C$ **Ans.**

Ex7 .Evaluate

$$\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$$

Solution. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

$$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x$$

Writing $z = \sec y$, so that $\frac{dz}{dx} = \sec y \tan y \frac{dy}{dx}$

The equation becomes $\frac{dz}{dx} + z \tan x = \cos^2 x$

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

$\therefore$ The solution of the equation is

$$z \sec x = \int \cos^2 x \sec x\, dx + C$$

$$\sec y \sec x = \int \cos x\, dx + C = \sin x + C$$

$$\sec y = (\sin x + C) \cos x$$ **Ans.**

Ex 8.Evaluate

$$x\left[\frac{dx}{dy} + y\right] = 1 - y$$

Solution

which is in linear form of $\frac{dy}{dx} + Py = Q$.

$\therefore$ $P = \left(1 + \frac{1}{x}\right)$, $Q = \frac{1}{x}$

$$\text{I.F.} = e^{\int P dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x+\log x} = e^x . e^{\log x} = e^x . x = x e^x$$

$$y(\text{I.F.}) = \int \text{I.F.}(Q\, dx) + C$$

$$y(x.e^x) = \int (x.e^x) \times \frac{1}{x} dx + C \Rightarrow y(x.e^x) = \int e^x dx + C$$

$$y\,(x\,.\,e^x) = e^x + C$$

$$y = \frac{1}{x} + \frac{C}{x} e^{-x}$$ **Ans.**

Ex 9.Evaluate

$y \log y \, dx + (x - \log y) \, dy = 0$

Solution. We have,

$$y \log y \, dx + (x - \log y) \, dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-x + \log y}{y \log y} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{-x}{y \log y} + \frac{\log y}{y \log y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$

$$\text{I.F.} = e^{\int \frac{1}{y \log y} dy} = e^{\log(\log y)} = \log y$$

Its solution is $\quad x.\log y = \int \frac{1}{y} (\log y) \, dy$

$$x.\log y = \frac{(\log y)^2}{2} + C$$ **Ans.**

Ex 10.Evaluate

$(1 + y^2) \, dx = (\tan^{-1} y - x) \, dy.$

Solution. $(1 + y^2) \, dx = (\tan^{-1} y - x) \, dy$

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2} \quad \Rightarrow \quad \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

This is a linear differential equation.

$$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Its solution is $\quad x.e^{\tan^{-1} y} = \int e^{\tan^{-1} y} \frac{\tan^{-1} y}{1 + y^2} dy + C$

Put $\quad \tan^{-1} y = t$ on R.H.S., so that $\frac{1}{1+y^2} dy = dt$

$$x.e^{\tan^{-1} y} = \int e^t . t \, dt + C = t.e^t - e^t + C = e^{\tan^{-1} y} \left(\tan^{-1} y - 1\right) + C$$

$$x = \left(\tan^{-1} y - 1\right) + Ce^{-\tan^{-1} y}$$ **Ans.**

Ex 11.Evaluate

$$r \sin \theta - \frac{dr}{d\theta} \cos \theta = r^2$$

Solution. The given equation can be written as $-\dfrac{dr}{d\theta}\cos\theta + r\sin\theta = r^2$ ... (1)

Dividing (1) by $r^2\cos\theta$, we get $-r^{-2}\dfrac{dr}{d\theta} + r^{-1}\tan\theta = \sec\theta$ ... (2)

Putting $r^{-1} = v$ so that $-r^{-2}\dfrac{dr}{d\theta} = \dfrac{dv}{d\theta}$ in (2), we get

$$\frac{dv}{d\theta} + v\tan\theta = \sec\theta$$

$$\text{I.F.} = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$$

Solution is $v\sec\theta = \int \sec\theta, \sec\theta + C \Rightarrow v\sec\theta = \int \sec^2\theta\, d\theta + C$

$$\frac{\sec\theta}{r} = \tan\theta + C \Rightarrow r^{-1} = (\sin\theta + C\cos\theta)$$

$\therefore$ $r = \dfrac{1}{\sin\theta + C\cos\theta}$ **Ans.**

2.3 Summary

A basic type of differential equations in mathematics, linear differential equations are used to simulate a variety of biological, physical, and economic systems. This is a brief overview that covers the main ideas, categories, approaches, and illustrations.

2.4 Keywords

- Linear Differential Equation
- Order
- First-order
- Second-order
- Homogeneous Equation

2.5 Self Assessment questions

1. $\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{y} = 2xe^{-x}$ **Ans.** $e^x + x^2y + Cy = 0$
2. $3\dfrac{dy}{dx} + 3\dfrac{y}{x} = 2x^4y^4$ **Ans.** $\dfrac{1}{y^3} = x^5 + Cx^3$
3. $\dfrac{dy}{dx} = y\tan x - y^2\sec x$ **Ans.** $\sec x = (\tan x + C)\, y$

4. $\frac{dy}{dx} = 2y \tan x + y^2 \tan^2 x$, if $y = 1$ at $x = 0$ **Ans.** $\frac{1}{y}\sec^2 x = -\frac{\tan^3 x}{3} + 1$

5. $\frac{dy}{dx} + \tan x \tan y = \cos x \sec y$ **Ans.** $\sin y \sec x = x + C$

6. $dy + y \tan x \,.\, dx = y^2 \sec x \,.\, dx$ **Ans.** $y (x + C) + \cos x = 0$

7. $(x^2 y^2 + xy) y\, dx + (x^2 y^2 - 1) x\, dy = 0$ **Ans.** $x y = \log C y$

8. $(x^2 + y^2 + x)\, dx + xy\, dy = 0$ **Ans.** $x^2 y^2 = -\frac{x^4}{2} - \frac{2x^3}{3} + C$

9. $\frac{dy}{dx} + y = 3e^x y^3$ **Ans.** $\frac{1}{y^2} = 6e^x + Ce^{2x}$

10. $(x - y^2)\, dx + 2\, x\, y\, dy = 0$ **Ans.** $\frac{y^2}{x} + \log x = C$

2.6 Case Study

Rhythmic Mass, spring, and Damper Mechanism

Imagine you have a mass (m), a spring (k), and a damper (c) with a damping coefficient. This is a mass-spring-damper system. An external force (F(t)) causes the mass to shift from its equilibrium position by a distance (x(t)).

1. Determine the differential equation controlling the mass's motion by analyzing its motion.
2. Determine the specific solution when (F(t)), using a variety of techniques such variable parameters or unknown coefficients.

2.7 References

1. Kristensson, G. (2020). Second Order Differential Equations: Special Functions and Their Classification. Germany: Springer New York.
2. Keskin, A. Ü. (2018). Ordinary Differential Equations for Engineers: Problems with MATLAB Solutions. Germany: Springer International Publishing.

Unit-3

Exact Differential Equations

Learning Objectives:

- To understand Exact Differential equations
- To understand reducible to the exact equations
- To understand variable separable differentiation

Structure:

3.1 Exact Differential Equations

3.2 Equation reducible to the exact equations

3.3 Summary

3.4 Keywords

3.5 Self Assessment questions

3.6 Case Study

3.7 References

3.1 Exact Differential Equations:-

Differential equations that can be stated in the following form are considered exact differential equations, which are a subset of ordinary differential equations.

M(x,y)dx+N(x,y)dy=0

The equation is called "exact" if there exists a function F(x,y) such that:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex 1. Solve:

$$(5x^4 + 3x^2y^2 - 2xy^3)\, dx + (2x^3y - 3x^2y^2 - 5y^4)\, dy = 0$$

Solution. Here, $M = 5x^4 + 3x^2y^2 - 2xy^3$, $\quad N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2, \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Now $\int M\, dx + \int (\text{terms of } N \text{ is not containing } x)\, dy = C$ (y constant)

$$\int \left(5x^4 + 3x^2y^2 - 2xy^3\right) dx + \int -5y^4 dy = C$$

$\Rightarrow$ $x^5 + x^3y^2 - x^2y^3 - y^5 = C$ **Ans.**

Ex 2.Solve:

$$\left\{2xy\cos x^2 - 2xy + 1\right\} dx + \left\{\sin x^2 - x^2 + 3\right\} dy = 0$$

Solution. Here we have

$$\left\{2xy\cos x^2 - 2xy + 1\right\} dx + \left\{\sin x^2 - x^2 + 3\right\} dy = 0 \qquad \dots (1)$$

$$M\, dx + N\, dy = 0 \qquad \dots (2)$$

Comparing (1) and (2), we get

$$M = 2xy\cos x^2 - 2xy + 1 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$$

$$N = \sin x^2 - x^2 + 3 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x\cos x^2 - 2x$$

Here, $\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence solution is $\int\limits_{y \text{ as const}} M\,dx + \int$ (terms of N not containing x) $dy = C$

$$\int (2xy\cos x^2 - 2xy + 1)\,dx + \int 3\,dy = C$$

$$\Rightarrow \quad \int [y(2x\cos x^2) - y(2x) + 1]\,dx + 3\int dy = C$$

$$\Rightarrow \quad y\int 2x\cos x^2 dx - y\int 2x\,dx + \int 1\,dx + 3\int y\,dy = C$$

Put $x^2 = t$ so that $2x\,dx = dt$

$$y\int \cos t\,dt - 2y\frac{x^2}{2} + x + 3y = C$$

$$\Rightarrow \quad y\sin t - x^2 y + x + 3y = C$$

$$y\sin x^2 - yx^2 + x + 3y = C$$

Ans.

Ex 3 ***Solve:***

$$(1 + e^{x/y}) + e^{x/y}\left(1 - \frac{x}{y}\right)\frac{dy}{dx} = 0$$

$$N = e^{\frac{x}{y}} - e^{\frac{x}{y}}\frac{x}{y} \quad \Rightarrow \quad \frac{\partial N}{\partial x} = \frac{1}{y}e^{\frac{x}{y}} - \frac{1}{y}e^{\frac{x}{y}} - \frac{x}{y^2}e^{\frac{x}{y}} = -\frac{x}{y^2}e^{\frac{x}{y}}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given equation is exact.

Its solution is $\int \left(1 + e^{\frac{x}{y}}\right) dx + \int$ (terms of N not containing x) $dy = C$

$$\Rightarrow \quad \int \left(1 + e^{\frac{x}{y}}\right) dx + \int 0\,dy = C \quad \Rightarrow \quad x + ye^{\frac{x}{y}} = C$$

Ans.

Ex 4.*Solve:*

$$[1+\log(x\,y)]\,dx+\left[1+\frac{x}{y}\right]dy=0$$

Solution. $[1+\log x\,y]\,dx+\left[1+\frac{x}{y}\right]dy=0$

$\therefore \quad [1+\log x+\log y]\,dx+\left[1+\frac{x}{y}\right]dy=0$

which is in the form $M\,dx + N\,dy = 0$

$M = [1 + \log x + \log y]$ and $N=1+\frac{x}{y}$

$\Rightarrow \frac{\partial M}{\partial y}=\frac{1}{y}$ and $\Rightarrow \frac{\partial N}{\partial x}=\frac{1}{y}$ $\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Hence the given differential equation is exact.

$\therefore$ Solution is $\int_{y\text{ constant}} M\,dx+\int N\,(\text{terms not containing } x)\,dy=C$

y constant

$\therefore \quad \int(1+\log x+\log y)\,dx+\int dy=C$

$\Rightarrow \quad x+\int \log x\,dx+\int \log y\,dx+y=C$ ... (1)

Now, $\int \log x\,dx=\int \log x.(1)\,dx = (\log x)x-\int\left[\frac{d}{dx}(\log x)x\right]dx=x\log x-\int\frac{1}{x}.x\,dx$

$=x\log x-\int dx=x\log x-x=x[\log x-1]$

$\therefore$ Equation (1) becomes $\Rightarrow$ $x + x\log x - x + x\log y + y = C$

$x\,[\log x + \log y] + y = C \Rightarrow x\log xy + y = C$ **Ans.**

Exercise

Solve:

1. $(x + y - 10)\,dx + (x - y - 2)\,dy = 0$ **Ans.** $\frac{x^2}{2}+xy-10x-\frac{y^2}{2}-2y=C$

2. $(y^2 - x^2)\,dx + 2x\,y\,dy = 0$ **Ans.** $\frac{x^3}{3}=x\,y^2+C$

3. $\left(1+3e^{x/y}\right)dx+3e^{x/y}\left(1-\frac{x}{y}\right)dy=0$ **Ans.** $x + 3y\,e^{x/y} = C$

3.2 Equation reducible to the exact equations:-

Rule 1. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ is a function of x alone, say $f(x)$, then $\text{I.F.} = e^{\int f(x)dx}$

Example-5: ***Solve***

$(2x \log x - xy)\, dy + 2y\, dx = 0$

Solution. $M = 2y, \qquad N = 2x \log x - xy \qquad \dots (1)$

$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 2(1+\log x) - y$$

Here, $$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N} = \frac{2-2-2\log x + y}{2x\log x - xy} = \frac{-(2\log x - y)}{x(2\log x - y)} = -\frac{1}{x} = f(x)$$

$$\text{I.F.} = e^{\int f(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

On multiplying the given differential equation (1) by $\frac{1}{x}$, we get

$$\frac{2y}{x}dx + (2\log x - y)dy = 0 \quad \Rightarrow \quad \int \frac{2y}{x}dx + \int -y\, dy = c$$

$$\Rightarrow \quad 2y\log x - \frac{1}{2}y^2 = c \qquad \textbf{Ans.}$$

Exercise

Solve:

1. $(y \log y)\, dx + (x - \log y)\, dy = 0$ **Ans.** $2x \log y = c + (\log y)^2$
2. $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(1+y^2)x\, dy = 0$ **Ans.** $\frac{yx^4}{4} + \frac{y^3x^4}{12} + \frac{x^6}{12} = c$
3. $(y - 2x^3)\, dx - x(1 - xy)\, dy = 0$ **Ans.** $-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$
4. $(x \sec^2 y - x^2 \cos y)\, dy = (\tan y - 3x^4)\, dx$ **Ans.** $-\frac{1}{x}\tan y - x^3 + \sin y = c$
5. $(x - y^2)\, dx + 2xy\, dy = 0$ **Ans.** $y^2 = cx - x \log x$

Rule II. If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ is a function of y alone, say $f(y)$, then

$$\text{I.F.} = e^{\int f(y)dy}$$

Ex 6: ***Solve***

$(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0$

Solution. Here $M = y^4 + 2y;$ $\qquad N = xy^3 + 2y^4 - 4x$ $\qquad$ ...(1)

$$\therefore \qquad \frac{\partial M}{\partial y} = 4y^3 + 2; \qquad \frac{\partial N}{\partial x} = y^3 - 4$$

$$\therefore \qquad \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{(y^3 - 4) - (4y^3 + 2)}{y^4 + 2y} = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = -\frac{3}{y} = f(y)$$

$$\text{I.F.} = e^{\int f(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

On multiplying the given equation (1) by $\dfrac{1}{y^3}$ we get the exact differential equation.

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

$$\int \left(y + \frac{2}{y^2}\right)dx + \int 2y\, dy = c \qquad \Rightarrow \qquad x\left(y + \frac{2}{y^2}\right) + y^2 = c \qquad \textbf{Ans.}$$

Exercise

Solve:

1. $(3x^2y^4 + 2xy)\, dx + (2x^3y^3 - x^2)\, dy = 0$ $\qquad$ **Ans.** $x^3y^2 + \dfrac{x^2}{y} = c$

2. $(xy^3 + y)\, dx + 2(x^2y^2 + x + y^4)\, dy = 0$ $\qquad$ **Ans.** $\dfrac{x^2y^4}{2} + xy^2 + \dfrac{y^6}{3} = c$

3. $y(x^2y + e^x)dx - e^x dy = 0$ $\qquad$ **Ans.** $\dfrac{x^3}{3} + \dfrac{e^x}{y} = c$

4. $(2x^4y^4e^y + 2xy^3 + y)\, dx + (x^2y^4e^y - x^2y^2 - 3x)\, dy = 0$ $\qquad$ **Ans.** $x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = c$

Rule III. If M is of the form $M = y\, f_1(xy)$ and N is of the form $N = x\, f_2(xy)$

Then $\qquad \text{I.F.} = \dfrac{1}{M.x - N.y}$

Ex 7: ***Solve***

$y\,(xy + 2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0$

Dividing (1) by xy, we get

$$y\,(1 + 2xy)\,dx + x\,(1 - xy)\,dy = 0 \qquad \text{... (2)}$$

$$\text{M} = y\,f_1\,(xy), \quad \text{N} = x\,f_2\,(xy)$$

$$\text{I.F.} = \frac{1}{\text{M}x - \text{N}y} = \frac{1}{xy(1+2xy) - xy(1-xy)} = \frac{1}{3x^2y^2}$$

On multiplying (2) by $\frac{1}{3x^2y^2}$, we have an exact differential equation

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0 \quad \Rightarrow \quad \int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}dy = c$$

$$\Rightarrow \quad -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c \quad \Rightarrow \quad -\frac{1}{xy} + 2\log x - \log y = b$$ **Ans.**

Exercise

Solve:

1. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$ — **Ans.** $\log\left(\frac{x}{y}\right) - xy = A$

2. $y\,(1 + xy)\,dx + x(1 - xy)\,dy = 0$ — **Ans.** $xy\log\left(\frac{y}{x}\right) = c\,xy - 1$

2. $y\,(1 + xy)\,dx + x\,(1 + xy + x^2y^2)\,dy = 0$ — **Ans.** $\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = c$

4. $(xy\sin xy + \cos xy)\,y\,dx + (xy\sin xy - \cos xy)\,x\,dy = 0$ — **Ans.** $y\cos xy = cx$

Rule IV. For of this type of $x^m y^n (ay\,dx + bx\,dy) + x^{m'} y^{n'} (a'\,y\,dx + b'\,x\,dy) = 0$, the integrating factor is $x^h\,y^k$.

where $$\frac{m+h+1}{a} = \frac{n+k+1}{b}, \quad \text{and} \quad \frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

Ex 8: ***Solve***

$(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$

Solution. $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$

$$y^2\,(ydx + 2xdy) + x^2\,(-2ydx - xdy) = 0$$

Here $m = 0,\ h = 2,\ a = 1,\ b = 2,\quad m' = 2,\ n' = 0,\ a' = -2,\ b' = -1$

$$\frac{0+h+1}{1} = \frac{2+k+1}{2} \quad \text{and} \quad \frac{2+h+1}{-2} = \frac{0+k+1}{-1}$$

$\Rightarrow$ $2h + 2 = 2 + k + 1$ and $h + 3 = 2k + 2$

$\Rightarrow$ $2h - k = 1$ and $h - 2k = -1$

On solving $h = k = 1$. Integrating Factor $= xy$

Multiplying the given equation by xy, we get

$$(xy^4 - 2x^3y^2)\,dx + (2x^2y^3 - x^4y)\,dy = 0$$

which is an exact differential equation.

$$\int (xy^4 - 2x^3y^2)dx = C \quad \Rightarrow \quad \frac{x^2y^4}{2} - \frac{2x^4y^2}{4} = C$$

$\Rightarrow$ $x^2y^4 - x^4y^2 = C' \quad \Rightarrow \quad x^2y^2(y^2 - x^2) = C'$ **Ans.**

Ex 9.Solve

$(3y - 2xy^3)\,dx + (4x - 3x^2y^2)\,dy = 0.$

Solution. $(3y - 2xy^3)\,dx + (4x - 3x^2y^2)\,dy = 0$

$\Rightarrow$ $(3y\,dx + 4x\,dy) + xy^2(-2y\,dx - 3x\,dy) = 0$ ...(1)

Comparing the coefficients of (1) with

$x^m y^n (a\,y\,dx + b\,x\,dy) + x^{m'} y^{n'} (a'\,y\,dx + b'\,x\,dy) = 0,$ we get

$m = 0,\ n = 0,\ a = 3,\ b = 4$

$m' = 1,\ n' = 2,\ a' = -2,\ b' = -3$

To find the integrating factor $x^h y^k$

$$\frac{m+h+1}{a} = \frac{n+k+1}{b} \quad \text{and} \quad \frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

$$\frac{0+h+1}{3} = \frac{0+k+1}{4} \quad \text{and} \quad \frac{1+h+1}{-2} = \frac{2+k+1}{-3}$$

$\Rightarrow$ $\frac{h+1}{3} = \frac{k+1}{4}$ and $\frac{h+2}{2} = \frac{k+3}{3} \Rightarrow 4h - 3k + 1 = 0$ ... (2)

and $3h - 2k = 0 \quad \Rightarrow \quad h = \frac{2k}{3}$ ... (3)

Putting the value of h from (3) in (2), we get

$$\frac{8k}{3} - 3k + 1 = 0 \quad \Rightarrow \quad -\frac{k}{3} + 1 = 0 \quad \Rightarrow \quad k = 3$$

Putting $k = 3$ in (2), we get $h = \frac{2k}{3} = \frac{2\times 3}{3} = 2$

$$\text{I.F.} = x^h y^k = x^2 y^3$$

On multiplying the given differential equation by x^2y^3, we get

$$x^2y^3(3y - 2xy^3)dx + x^2y^3(4x - 3x^2y^2)\,dy = 0$$

$$(3x^2y^4 - 2x^3y^6)\,dx + (4x^3y^3 - 3x^4y^5)\,dy = 0$$

This is the exact differential equation.

Its solution is $\int (3x^2y^4 - 2x^3y^6)dx = 0 \quad \Rightarrow \quad x^3y^4 - \frac{x^4}{2}y^6 = C$ **Ans.**

Exercise

Solve:

1. $(2y\,dx + 3x\,dy) + 2xy\,(3y\,dx + 4x\,dy) = 0$ **Ans.** $x^2y^3(1 + 2xy) = c$
2. $(y^2 + 2yx^2)\,dx + (2x^3 - xy)\,dy = 0$ **Ans.** $4(xy)^{1/2} - \frac{2}{3}\left(\frac{y}{x}\right)^{3/2} = c$
3. $(3x + 2y^2)y\,dx + 2x\,(2x + 3y^2)\,dy = 0$ **Ans.** $x^2y^4(x + y^2) = c$
4. $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0$ **Ans.** $\frac{7}{5}x^{10/7}y^{-5/7} - \frac{7}{4}x^{-4/7}y^{-12/7} = c$
5. $x\,(3y\,dx + 2x\,dy) + 8y^4\,(y\,dx + 3x\,dy) = 0$ **Ans.** $x^3y^2 + 4x^2y^6 = c$

Rule V.

If the given equation $M\,dx + N\,dy = 0$ is homogeneous equation and $Mx + Ny \neq 0$, then $\frac{1}{Mx + Ny}$ is an integrating factor.

Ex 10.Solve

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$

Solution. $(x^3 + y^3)\,dx - (xy^2)\,dy = 0$ ... (1)

Here $M = x^3 + y^3, \quad N = -xy^2$

$$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x(x^3 + y^3) - xy^2(y)} = \frac{1}{x^4}$$

Multiplying (1) by $\frac{1}{x^4}$ we get $\frac{1}{x^4}(x^3 + y^3)dx + \frac{1}{x^4}(-xy^2)dy = 0$

$\Rightarrow \quad \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}dy = 0$, which is an exact differential equation.

$$\int \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx = c \quad \Rightarrow \quad \log x - \frac{y^3}{3x^3} = c \quad \textbf{Ans.}$$

3.3 Summary

By identifying them as the differential of a function, exact differential equations belong to a unique family of differential equations that are quite simple to solve. This is a brief synopsis that covers the main ideas, approaches to solving the problem, and illustrations.

3.4 Keywords

- Exact differential equations
- Homogeneous Equation

3.5 Self Assessment questions

1. $x^2y\,dx - (x^3 + y^3)\,dy = 0$ **Ans.** $-\dfrac{x^3}{3y^3} + \log y = c$
2. $(y^3 - 3xy^2)\,dx + (2x^2y - xy^2)\,dy = 0$ **Ans.** $\dfrac{y}{x} + 3\log x - 2\log y = c$
3. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ **Ans.** $\dfrac{x}{y} - 2\log x + 3\log y = c$
4. $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$ **Ans.** $x^2y^4 - x^4y^2 = c$

3.6 Case Study

The Use of Heat Transfer in Engineering Design As an engineer, you can be assigned the responsibility of creating a cooling system for a high-performing electrical gadget, such a high-power laser or a CPU. To avoid overheating and guarantee optimum performance and lifespan, it is essential to comprehend how heat drains from the device during the design phase.

Question: Your goal is to optimize the cooling system design and simplify the analysis by modeling the heat transfer process with differential equations and change of variables.

3.7 References

1. "Elementary Differential Equations and Boundary Value Problems" by William E. Boyce and Richard C. DiPrima
2. "Advanced Engineering Mathematics" by Erwin Kreyszig

Unit-4

Differential Equations of first order and higher degree

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and higher degree

Structure

4.1 Differential Equations (DEs) of first order and higher degree:-

The DE which involve $\frac{dy}{dx}$ and denoted by p and form $f(x, y, p)$.

Case 1. Equation solvable for p

Ex. 1: Solve $x^2 = 1 + p^2$

Solution. $x^2 = 1 + p^2 \quad \Rightarrow \quad p^2 = x^2 - 1$

$$\Rightarrow \quad p = \pm\sqrt{x^2 - 1} \quad \Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{x^2 - 1} \quad \Rightarrow \quad dy = \pm\sqrt{x^2 - 1}\, dx$$

which gives on integration $y = \pm\frac{x}{2}\sqrt{x^2 - 1} \mp \frac{1}{2}\log\left(x + \sqrt{x^2 - 1}\right) + c$ **Ans.**

Case 2. Equation solvable for y

Ex 2 :*Solve* $y = (x - a)p - p^2$

Differentiating (1) w.r.t. "x" we obtain

$$\frac{dy}{dx} = p + (x - a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$p = p + (x - a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow \quad 0 = (x - a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow \quad 0 = \frac{dp}{dx}[x - a - 2p] \quad \Rightarrow \quad \frac{dp}{dx} = 0$$

On integration, we get $p = c$.

Putting the value of p in (1), we get

$$y = (x - a)\, c - c^2$$ **Ans.**

Case 3. Equation solvable for x

Ex 3: *Solve* $y = 2px + yp^2$

Solution. $y = 2px + yp^2$ ... (1)

$\Rightarrow$ $2px = y - yp^2$ $\Rightarrow$ $2x = \frac{y}{p} - yp$ ... (2)

Differentiating (2) w.r.t. "y" we get

$$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy}$$

$\Rightarrow$ $\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy}$ $\Rightarrow$ $\frac{1}{p} + p = -\frac{y}{p^2}\frac{dp}{dy} - y\frac{dp}{dy}$

$\Rightarrow$ $\frac{1}{p} + p = -y\left(\frac{1}{p^2} + 1\right)\frac{dp}{dy}$ $\Rightarrow$ $\frac{1+p^2}{p} = -y\frac{1+p^2}{p^2}\frac{dp}{dy}$

$\Rightarrow$ $1 = -\frac{y}{p}\frac{dp}{dy}$ $\Rightarrow$ $-\frac{dy}{y} = \frac{dp}{p}$ $\Rightarrow$ $-\log y = \log p + \log c'$

$\Rightarrow$ $\log p\, y = \log c$ $\Rightarrow$ $p\, y = c$ $\Rightarrow$ $p = \frac{c}{y}$

Putting the value of p in (1), we get

$$y = 2\left(\frac{c}{y}\right)x + y\left(\frac{c^2}{y^2}\right) \Rightarrow y^2 = 2\,cx + c^2$$

$\Rightarrow$ $y^2 = c(2x + c)$ **Ans.**

Exercise

Solve:

1. $xp^2 + x = 2yp$ **Ans.** $2cy = c^2x^2 + 1$
2. $x(1 + p^2) = 1$ **Ans.** $y - c = \sqrt{(x - x^2)} - \tan^{-1}\sqrt{\frac{1-x}{x}}$
3. $x^2p^2 + xyp - 6y^2 = 0$ **Ans.** $y = \frac{c}{x^3}, y = c_1x^2$
4. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ **Ans.** $xy = c,\ x^2 - y^2 = c$
5. $y = px + p^3$ **Ans.** $y = ax + a^3$
6. $x^2\,(y - px) = yp^2$ **Ans.** $y^2 = cx^2 + c^2$

4.2 Orthogonal Trajectories:-

Working rule to find orthogonal trajectories of curves

Step 1. By differenciating the equaton of curves find the differential equations in the form $f\left(x, y, \frac{dy}{dx}\right) = 0$

Step 2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ $(M_1.\,M_2 = -1)$

Step 3. Solve the differential equation of the orthogonal trajectories *i.e.*, $f\left(x, y - \frac{dx}{dy}\right) = 0$

Ex 4: Find the orthogonal trajectories of curves xy=c

Solution. Here, we have

$$xy = c \quad \text{... (1)}$$

Differentiating (1), w.r.t., "x", we get

$$y + x\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

On replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$\Rightarrow \quad -\frac{dx}{dy} = -\frac{y}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx \quad \text{... (2)}$$

Integrating (2), we get $\frac{y^2}{2} = \frac{x^2}{2} + c$

$$\Rightarrow \quad y^2 - x^2 = 2c \quad \textbf{Ans.}$$

Ex 5: Solve that the family of parabolas $y^2 = 2cx + c^2$ is self orthogonally

Solution. Here we have

$$y^2 = 2cx + c^2 \quad \text{... (1)}$$

Differentiating (1), we get $2y\frac{dy}{dx} = 2c \quad \Rightarrow \quad c = y\frac{dy}{dx}$

Putting the value of c in (1), we have $y^2 = 2\left(y\frac{dy}{dx}\right)x + \left(y\frac{dy}{dx}\right)^2$ ... (2)

Putting $\frac{dy}{dx} = p$ in (2), we get

$$y^2 = 2ypx + y^2p^2 \quad \text{... (3)}$$

This is differential equation of give n family of parabolas.

For orthogonal trajectories we put $-\frac{1}{p}$ for p in (3)

$$y^2 = 2y\left(-\frac{1}{p}\right)x + y^2\left(-\frac{1}{p}\right)^2 \quad \Rightarrow \quad y^2 = -\frac{2yx}{p} + \frac{y^2}{p^2}$$

$$\Rightarrow \quad y^2p^2 = -2pyx + y^2$$

Rewriting, we get

$$\Rightarrow \quad y^2 = 2ypx + y^2p^2$$

Which is same as equation (3). Thus (2) is D.E. for the given family and its orthogonal trajectories.

Hence, the given family is self-orthogonal. **Proved.**

4.3 Wronskian and its properties:-

Remember that we used a fundamental set of two solutions of second order linear homogeneous ODE in Standard form to derive the Wronskian and the concept of linearly independent functions:

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are both continous on some interval I.

In order to verify the linear independence of the two solutions to the aforementioned equations, we use the Wronskian. It turns out that looking for just two fundamental solutions to the aforementioned ODE is not the whole meaning of linear independence. Additionally, not just two solutions of the ODE but any two differentiable functions can have their linear independence confirmed by a nonzero Wronskian..

Linear Independence & the Wronskian for any two functions

Remember how we defined the linear dependence of two functions, f and g, on an interval that is open. I: If there are constants c_1 and c_2, which are not zero, then f and g are linearly dependent on each other.

$$c_1 f(t) + c_2 g(t) = 0, \quad \text{for all } t \in I$$

If we choose $c_1 = 0 = c_2$, then we say f and g are linearly independent. Since we are just thinking about two functions, in this particular example, linear dependence is equal to one function being a scalar multiple of the other:

$f(t) = Cg(t)$ **or** $g(t) = Cf(t)$ **for some constant** C.

Note that C may be zero.

Linear Independence & the Wronskian for two solutions of ODE

If we are considering $f = y_1$ and $g = y_2$ to be two solutions of ODE

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are both continuous on some interval I, then the Wronskian has some extra properties which are given by Abel's Theorem:

$$W[y_1, y_2](t) = ce^{-\int p(t)dt}, \quad \text{for some constant } c.$$

Fundamentally, this theorem states that if two ODE solutions are linearly independent, then their Wronskian is never zero on interval I, that is, $c \neq 0$. If not, there is always a zero Wronskian, meaning that $c=0$, and the solutions are linearly dependant. The primary outcome that we believe is helpful in determining a basic pair of solutions for a linear homogeneous differential equation of second order is this one.

4.4 Summary

Differential equations of first order and higher degree encompass a broad category in mathematics with significant applications across various fields. Here's a summary:

First-order differential equations: The derivatives of a function with respect to a single independent variable are involved in these equations. They can be classified into various types, including:

- **Ordinary Differential Equations (ODEs):** Involving one independent variable.
- **Partial Differential Equations (PDEs):** Involving multiple independent variables.

1. **Higher-order differential equations:** These equations involve derivatives of a function with respect to one or more independent variables, where the highest derivative present is of order greater than one.

4.5 Keywords

- Higher-order differential equations
- Linear independence
- Wronskian

4.6 Self Assessment questions

1. $y^2 = cx^3$ **Ans.** $(x+1)^2 + y^2 = a^2$

2. $x^2 - y^2 = cx$ **Ans.** $y(y^2 + 3x^2) = c$

3. $x^2 - y^2 = c$ **Ans.** $xy = c$

4. $(a+x)y^2 = x^2(3a - x)$ **Ans.** $(x^2 + y^2)^5 = cy^3(5x^2 + y^2)$

5. $y = ce^{-2x} + 3x$, passing through the point (0, 3) **Ans.** $9x - 3y + 5 = -4e^{6(3-y)}$

6. $16x^2 + y^2 = c$ **Ans.** $y^{16} = kx$

7. $y = \tan x + c$ **Ans.** $2x + 4y + \sin 2x = k$

8. $y = ax^2$ **Ans.** $x^2 + 2y^2 = c$

4.7 Case Study

Improving Customer Satisfaction in a Restaurant

Imagine you're the manager of a restaurant facing a decline in customer satisfaction despite serving delicious food. You've identified that long wait times and inconsistent service are the main issues.

Question: Your restaurant's current operational model relies heavily on fixed procedures and limited flexibility in handling customer needs. This rigidity leads to dissatisfaction when customers' preferences or needs aren't met promptly.

4.8 References

1. "Elementary Differential Equations and Boundary Value Problems" by William E. Boyce and Richard C. DiPrima
2. "Advanced Engineering Mathematics" by Erwin

Unit-5

Linear Homogeneous Differential Equations with Constant Coefficients

Learning Objectives:

- To understand Linear Homogeneous Differential equations
- To understand method of complementary function
- To understand rule of particular integral

Structure

5.1 Linear Homogeneous Differential Equations of second order with Constant

Coefficients:-

The General form of D.E of second Order is given by

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where Pand Q are constants and R is a function of x and D is diffential operator.

$$Dy = \frac{dy}{dx}, \quad D^2y = \frac{d^2y}{dx^2}$$

$\frac{1}{D}$ stands for the operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ can be written in the operator form.

$$D^2y + P\,Dy + Q\,y = R \qquad \Rightarrow \qquad (D^2 + PD + Q)\,y = R$$

C**omplete solution = complementary function + particular integral**

Let us consider a L.D.E of first order

$$\frac{dy}{dx} + Py = Q \qquad \text{...(1)}$$

Its solution is $\quad ye^{\int P dx} = \int (Q\,e^{\int P dx})\,dx + C$

$\Rightarrow \quad y = Ce^{-\int P dx} + e^{-\int P dx}\int (Qe^{\int P dx})\,dx$

$\Rightarrow \quad y = cu + v\text{(say)} \qquad \text{...(2)}$

where $u = e^{-\int P dx}$ and $v = e^{-\int P dx}\int Q\,e^{\int P dx}dx$

(i) Now differentiating $u = e^{-\int P dx}$ w.r.t. x, we get $\frac{du}{dx} = -Pe^{-\int P dx} = -Pu$

$\Rightarrow \qquad \frac{du}{dx} + Pu = 0 \qquad \Rightarrow \qquad \frac{d(cu)}{dx} + P(cu) = 0$

which shows that $y = c.u$ is the solution of $\frac{dy}{dx} + Py = 0$

(ii) Differentiating $v = e^{-\int P dx} \int (Qe^{\int P dx}) dx$ with respect to x, we get

$$\frac{dv}{dx} = -Pe^{\int P dx} \int (Qe^{\int P dx}) dx + e^{-\int P dx} Qe^{\int P dx} \qquad \Rightarrow \qquad \frac{dv}{dx} = -Pv + Q$$

$\Rightarrow \qquad \frac{dv}{dx} + Pv = Q$ which shows that $y = v$ is the solution of $\boxed{\frac{dy}{dx} + Py = Q}$

$$\Rightarrow \qquad \boxed{y = C.F. + P.I.}$$

Rules for complementery function

(1) In finding the complementary function, R.H.S. of the given equation is replaced by zero.

(2) Let $y = C_1 e^{mx}$ be the C.F. of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad ...(1)$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) then $C_1 e^{mx} (m^2 + Pm + Q) = 0$

$\Rightarrow \qquad m^2 + Pm + Q = 0$. It is called **Auxiliary equation**.

(3) Solve the auxiliary equation :

Case I : Roots, Real and Different. If m_1 and m_2 are the roots, then the C.F. is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Case II : Roots, Real and Equal. If both the roots are m_1, m_1 then the C.F. is

$$y = (C_1 + C_2 x) e^{m_1 x}$$

Equation (1) can be written as

$$(D - m_1)(D - m_1)y = 0 \qquad ... (2)$$

Replacing $(D - m_1)y = v$ in (2), we get

$$(D - m_1)v = 0 \qquad ... (3)$$

$\frac{dv}{dx} - m_1 v = 0 \qquad \Rightarrow \qquad \frac{dv}{v} = m_1 dx \qquad \Rightarrow \qquad \log v = m_1 x + \log c_2 \qquad \Rightarrow \qquad v = c_2 e^{m_1 x}$

$$v = c_2 e^{m_1 x}$$

$$v = c_2 e^{m_1 x}$$

From (3) $$(D-1)y = c_2 e^{m_1 x}$$

This is the linear differential equation.

$$\text{I.F.} = e^{-m_1 \int dx} = e^{-m_1 x}$$

Solution is

$$y.e^{-m_1 x} = \int (c_2 e^{m_1 x})(e^{-m_1 x})\,dx + c_1 = \int c_2\,dx + c_1 = c_2 x + c_1$$

$$y = (c_2 x + c_1) e^{m_1 x}$$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{m_1 x}$$

Ex 1: Solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Solution. Given equation can be written as

$$(D^2 - 8D + 15)\, y = 0$$

Here auxiliary equation is $m^2 - 8m + 15 = 0$

$\Rightarrow$ $(m-3)(m-5) = 0$ $\therefore\ m = 3, 5$

Hence, the required solution is

$$y = C_1 e^{3x} + C_2 e^{5x}$$ **Ans.**

Ex 2: Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

$$y = 2 \text{ and } \frac{dy}{dx} = \frac{d^2y}{dx^2} \text{ when } x = 0.$$

Solution. Here the auxiliary equation is

$$m^2 + 4m + 5 = 0$$

Its root are $-2 \pm i$

The complementary function is

$$y = e^{-2x}(A\cos x + B\sin x) \qquad \text{...(1)}$$

On putting $y = 2$ and $x = 0$ in (1), we get

$$2 = A$$

On putting $A = 2$ in (1), we have

$$y = e^{-2x}[2\cos x + B\sin x] \qquad \text{...(2)}$$

On differentiating (2), we get

$$\frac{dy}{dx} = e^{-2x}[-2\sin x + B\cos x] - 2e^{-2x}[2\cos x + B\sin x]$$

$$= e^{-2x}[(-2B-2)\sin x + (B-4)\cos x]$$

$$\frac{d^2y}{dx^2} = e^{-2x}[(-2B-2)\cos x - (B-4)\sin x]$$

$$- 2e^{-2x}[(-2B-2)\sin x + (B-4)\cos x]$$

$$= e^{-2x}[(-4B+6)\cos x + (3B+8)\sin x]$$

But $$\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

e^{-2x} [(–2B –2) sin x + (B – 4) cos x] = e^{-2x} [(– 4B + 6) cos x + (3B + 8) sin x]

On putting $x = 0$, we get

$$B - 4 = -4B + 6 \quad \Rightarrow \quad B = 2$$

(2) becomes, $$y = e^{-2x}[2\cos x + 2\sin x]$$

$$y = 2e^{-2x}[\sin x + \cos x]$$ **Ans.**

Exercise

Solve

1. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$ **Ans.** $y = (C_1 + C_2x)\,e^{4x}$

2. $\frac{d^2y}{dx^2} + \mu^2 y = 0$ **Ans.** $y = C_1\cos\mu x + C_2\sin\mu x$

5.2 Rules for particular integral

(*i*) $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ If $f(a) = 0$ then $\frac{1}{f(D)}\cdot e^{ax} = x\cdot\frac{1}{f'(a)}\cdot e^{ax}$

If $f'(a) = 0$ then $\frac{1}{f(D)}\cdot e^{ax} = x^2\frac{1}{f''(a)}\cdot e^{ax}$

(*ii*) $\frac{1}{f(D)}x^n = [f(D)]^{-1}x^n$ Expand $[f(D)]^{-1}$ and then operate.

(*iii*) $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$ and $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$

If $f(-a^2) = 0$ then $\frac{1}{f(D^2)}\sin ax = x\cdot\frac{1}{f'(-a^2)}\cdot\sin ax$

(*iv*) $\frac{1}{f(D)}e^{ax}\cdot\phi(x) = e^{ax}\cdot\frac{1}{f(D+a)}\phi(x)$

(*v*) $\frac{1}{D+a}\phi(x) = e^{-ax}\int e^{ax}\cdot\phi(x)\,dx$

$$\boxed{\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}.}$$

We know that, $D.e^{ax} = a.e^{ax}$, $D^2e^{ax} = a^2.e^{ax}, \ldots\ldots\ldots\ldots, D^n e^{ax} = a^n e^{ax}$

Let $f(D)\, e^{ax} = (D^n + K_1D^{n-1} + \ldots + K_n)\, e^{ax} = (a^n + K_1a^{n-1} + \ldots + K_n)e^{ax} = f(a)\, e^{ax}$.

Operating both sides by $\frac{1}{f(D)}$

$$\frac{1}{f(D)} \cdot f(D) e^{ax} = \frac{1}{f(D)} \cdot f(a) e^{ax}$$

$$\Rightarrow \quad e^{ax} = f(a)\frac{1}{f(D)} \cdot e^{ax} \quad \Rightarrow \quad \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$$

If $f(a) = 0$, then the above rule fails.

Then $\frac{1}{f(D)}e^{ax} = x \cdot \frac{1}{f'(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax} \quad \Rightarrow \quad \boxed{\frac{1}{f(D)}e^{ax} = x\,.\,\frac{1}{f'(a)}e^{ax}}$

If $f'(a) = 0$ then $\boxed{\frac{1}{f(D)}e^{ax} = x^2 \frac{1}{f''(a)}e^{ax}}$

Ex 3: Solve

$$\frac{d^2x}{dt^2} + \frac{g}{t}x = \frac{g}{l}L$$

where g, l, L are constants subject to the conditions,

$$x = a, \ \frac{dx}{dt} = 0 \ \ at\ t = 0.$$

Solution. We have, $\frac{d^2x}{dt^2} + \frac{g}{t}x = \frac{g}{l}L \quad \Rightarrow \quad \left(D^2 + \frac{g}{l}\right)x = \frac{g}{l}L$

A.E. is $m^2 + \frac{g}{l} = 0 \quad \Rightarrow \quad m = \pm\, i\sqrt{\frac{g}{l}}$

$$\text{C.F.} = C_1 \cos\sqrt{\frac{g}{l}}\,t + C_2 \sin\sqrt{\frac{g}{l}}\,t$$

$$\text{P.I.} = \frac{1}{D^2 + \frac{g}{l}} \cdot \frac{g}{l}L = \frac{g}{l}L\frac{1}{D^2 + \frac{g}{l}}e^{0t} = \frac{g}{l}L\frac{1}{0 + \frac{g}{l}} = L \qquad [D = 0]$$

$\therefore$ General solution is = C.F. + P.I.

$$x = C_1 \cos\left(\sqrt{\frac{g}{l}}\right)t + C_2 \sin\left(\sqrt{\frac{g}{l}}\right)t + L \qquad \ldots(1)$$

$$\frac{dx}{dt} = -C_1\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}\right)t + C_2\sqrt{\frac{g}{l}}\cos\left(\sqrt{\frac{g}{l}}\right)t$$

Put $t = 0$ and $\frac{dx}{dt} = 0$

$$0 = C_2\sqrt{\frac{g}{l}} \qquad \therefore \quad C_2 = 0$$

(1) becomes $\qquad x = C_1 \cos\sqrt{\frac{g}{l}}\, t + L \qquad$ ...(2)

Put $x = a$ and $t = 0$ in (2), we get

$$a = C_1 + L \qquad \text{or} \qquad C_1 = a - L$$

On putting the value of C_1 in (2), we get $x = (a - L)\cos\left(\sqrt{\frac{g}{l}}\right)t + L$ **Ans.**

Ex 6: Solve

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$

Solution. $\qquad (D^2 + 6D + 9)y = 5e^{3x}$

Auxiliary equation is $m^2 + 6m + 9 = 0 \quad \Rightarrow \quad (m+3)^2 = 0 \quad \Rightarrow \quad m = -3, -3,$

$$\text{C.F.} = (C_1 + C_2x)\, e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 6D + 9}.5.e^{3x} = 5\frac{e^{3x}}{(3)^2 + 6(3) + 9} = \frac{5e^{3x}}{36}$$

The complete solution is $\qquad y = (C_1 + C_2x)e^{-3x} + \frac{5e^{3x}}{36}$ **Ans.**

Ex 7: ***Solve***

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$$

Solution. $\qquad (D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$

A.E. is $(m^2 - 6m + 9) = 0 \quad \Rightarrow \quad (m-3)^2 = 0, \quad \Rightarrow \quad m = 3, 3$

$$\text{C.F.} = (C_1 + C_2x)\,e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2 - 6D + 9}6e^{3x} + \frac{1}{D^2 - 6D + 9}7e^{-2x} + \frac{1}{D^2 - 6D + 9}(-\log 2)$$

$$= x\frac{1}{2D - 6}6e^{3x} + \frac{1}{4 + 12 + 9}7e^{-2x} - \log 2\frac{1}{D^2 - 6D + 9}e^{0x}$$

$$= x^2\frac{1}{2}\cdot 6\cdot e^{3x} + \frac{7}{25}e^{-2x} - \log 2\left(\frac{1}{9}\right) = 3x^2e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}\log 2$$

Complete solution is $y = (C_1 + C_2\, x)e^{3x} + 3x^2e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}\log 2$ **Ans.**

Exercise

Solve

1. $[D^2 + 5D + 6]\,[y] = e^x$ **Ans.** $C_2\, e^{-2x} + C_2 e^{-3x} + \frac{e^x}{12}$

2. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ **Ans.** $C_1 e^x + C_2 e^{2x} + \frac{e^{3x}}{2}$ *(A.M.I.E.T.E. June 2010, 2007)*

3. $(D^3 + 2D^2 - D - 2)\, y = e^x$ **Ans.** $C_1 e^x + C_2 e^{-x} + C_3 e^{-2x} + \frac{x}{6} e^x$

4. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sinh x$ **Ans.** $e^{-x}[C_1 \cos x + C_2 \sin x] + \frac{e^x}{10} - \frac{e^{-x}}{2}$

5. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$ **Ans.** $e^{-2x}(C_1 \cos x + C_2 \sin x) - \frac{1}{10}e^x - \frac{e^{-x}}{2}$

$$\boxed{\frac{1}{f(D)} x^n = [f(D)]^{-1} x^n.}$$

Ex 8 : ***Solve***

where a, R, p and l are constants subject to the conditions $y = 0,\ \frac{dy}{dx} = 0$ *at* $x = 0$.

Solution. $\frac{d^2y}{dx^2} + a^2 y = \frac{a^2}{p} R(l - x) \quad \Rightarrow \quad (D^2 + a^2)y = \frac{a^2}{p} R(l - x)$

A.E. is $m^2 + a^2 = 0 \Rightarrow m = \pm ia$

$$\text{C.F.} = C_1 \cos ax + C_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \frac{a^2}{p} R(l - x) = \frac{a^2 R}{p} \frac{1}{a^2} \left[\frac{1}{1 + \frac{D^2}{a^2}}\right](l - x) = \frac{R}{p}\left[1 + \frac{D^2}{a^2}\right]^{-1} (l - x)$$

$$= \frac{R}{p}\left[1 - \frac{D^2}{a^2}\right](l - x) = \frac{R}{p}(l - x)$$

$$y = C_1 \cos ax + C_2 \sin ax + \frac{R}{p}(l - x) \qquad \text{...(1)}$$

On differentiating (1), we get $\frac{dy}{dx} = -a\, C_1 \sin ax + a\, C_2 \cos ax - \frac{R}{p}$ ...(2)

On putting $\frac{dy}{dx} = 0$ and $x = 0$ in (2), we have

$$0 = a\, C_2 - \frac{R}{p} \quad \Rightarrow \quad C_2 = \frac{R}{a.p}$$

On putting the values of C_1 and C_2 in (1), we get

$$y = -\frac{R}{p} l \cos a.x + \frac{R}{a.p} \sin a.x + \frac{R}{p}(l - x) \Rightarrow y = \frac{R}{p}\left[\frac{\sin ax}{a} - l \cos a.x + l - x\right]$$ **Ans.**

Exercise $\frac{d^2y}{dx^2}+a^2y=\frac{a^2R}{p}(l-x)$

Solve

1. $(D^2+5D+4)\,y=3-2x$ **Ans.** $C_1e^{-x}+C_2e^{-4x}+\frac{1}{8}(11-4x)$

2. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x$ **Ans.** $(C_1+C_2x)\,e^{-x}+x-2$

3. $(2D^2+3D+4)\,y=x^2-2x$ **Ans.** $e^{-\frac{3}{4}x}[A\cos\frac{\sqrt{23}}{4}x+B\sin\frac{\sqrt{23}}{4}x]+\frac{1}{32}[8x^2-28x+13]$

$$\boxed{\frac{1}{f(D^2)}\sin ax=\frac{\sin ax}{f(-a^2)}} \qquad \boxed{\frac{1}{f(D^2)}\cdot\cos ax=\frac{\cos ax}{f(-a^2)}}$$

$D(\sin ax)=a.\cos ax$, $D^2(\sin ax)=D(a\cos ax)=-a^2.\sin ax$

$D^4(\sin ax)=D^2.D^2(\sin ax)=D^2(-a^2\sin ax)=(-a^2)^2\sin ax$

$(D^2)^n\sin ax=(-a^2)^n\sin ax$

Hence, $f(D^2)\sin ax=f(-a^2)\sin ax$

$$\frac{1}{f(D^2)}\cdot f(D^2)\sin ax=\frac{1}{f(D^2)}.f(-a^2).\sin ax$$

$$\sin ax=f(-a^2)\frac{1}{f(D^2)}\sin ax \quad\Rightarrow\quad \frac{1}{f(D^2)}\cdot\sin ax=\frac{\sin ax}{f(-a^2)}$$

Similarly, $$\frac{1}{f(D^2)}\cos ax=\frac{\cos ax}{f(-a^2)}$$

If $f(-a^2)=0$ then above rule fails.

$$\frac{1}{f(D^2)}\sin ax=x\frac{\sin ax}{f'(-a^2)}$$

If $f'(-a^2)=0$ then, $\frac{1}{f(D^2)}\sin ax=x^2\frac{\sin ax}{f''(-a^2)}$

Ex 9: Solve

$(D^2+4)\,y=\cos 2x$

Solution. $(D^2+4)\,y=\cos 2x$

Auxiliary equation is $m^2+4=0$

$m=\pm 2i$, C.F. $=A\cos 2x+B\sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4}\cos 2x = x.\frac{1}{2D}\cos 2x = \frac{x}{2}\left(\frac{1}{2}\sin 2x\right) = \frac{x}{4}\sin 2x$$

Complete solution is $y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$ **Ans.**

Ex 10: Solve

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^3} + 4\frac{dy}{dx} - 2y = e^x + \cos x$$

Solution. Given $(D^3 - 3D^2 + 4D - 2)\, y = e^x + \cos x$

A.E. is $m^3 - 3m^2 + 4m - 2 = 0$

$\Rightarrow$ $(m-1)(m^2 - 2m + 2) = 0$, *i.e.*, $m = 1,\ 1 \pm i$

$\therefore$ $\text{C.F.} = C_1e^x + e^x(C_2\cos x + C_3\sin x)$

$$\text{P.I.} = \frac{1}{(D-1)(D^2-2D+2)}e^x + \frac{1}{D^3-3D^2+4D-2}\cos x$$

$$= \frac{1}{(D-1)(1-2+2)}e^x + \frac{1}{(-1)D-3(-1)+4D-2}\cos x$$

$$= \frac{1}{(D-1)}e^x + \frac{1}{3D+1}\cos x = x\,\frac{1}{1}e^x + \frac{3D-1}{9D^2-1}\cos x$$

$$= e^x.x + \frac{(-3\sin x - \cos x)}{-9-1} = e^x.x + \frac{1}{10}(3\sin x + \cos x)$$

Hence, complete solution is

$$y = C_1e^x + e^x(C_2\cos x + C_3\sin x) + x\,e^x + \frac{1}{10}(3\sin x + \cos x) \qquad \textbf{Ans.}$$

Ex 11:Solve

Solution. $(D^3+1)y = \cos^2\left(\frac{x}{2}\right) + e^{-x}$

A.E. is $m^3 + 1 = 0$

$(m+1)(m^2 - m + 1) = 0 \quad \Rightarrow \quad m = -1$

or $m = \frac{-(-1) \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2} \quad \Rightarrow \quad m = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\therefore \quad \text{C.F.} = C_1e^{-x} + e^{\frac{x}{2}}\left[C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^3+1}\left[\cos^2\left(\frac{x}{2}\right) + e^{-x}\right] = \frac{1}{D^3+1}\cos^2\left(\frac{x}{2}\right) + \frac{1}{D^3+1}e^{-x} \quad [\text{Put } D = -1]$$

$$= \frac{1}{D^3+1}\left(\frac{1+\cos x}{2}\right) + \frac{1}{3D^2+1}e^{-x}$$

$$= \frac{1}{2}\frac{1}{D^3+1}e^{0x} + \frac{1}{2}\frac{1}{D^3+1}\cos x + \frac{1}{3(-1)^2+1}e^{-x} = \frac{1}{2} + \frac{1}{2}\frac{1}{-D+1}\cos x + \frac{1}{4}e^{-x}$$

$$= \frac{1}{2} - \frac{1}{2}\frac{(D+1)\cos x}{(D-1)(D+1)} + \frac{1}{4}e^{-x} = \frac{1}{2} - \frac{1}{2}\frac{(-\sin x + \cos x)}{(D^2-1)} + \frac{1}{4}e^{-x}$$

$$(D^3+1)y=\cos^2\left(\frac{x}{2}\right)+e^{-x}$$

$$=\frac{1}{2}+\frac{1}{2}\frac{\sin x}{(D^2-1)}-\frac{1}{2}\frac{1}{(D^2-1)}\cos x+\frac{1}{4}e^{-x}$$

Put $D^2=-1$ $=\frac{1}{2}+\frac{1}{2}\frac{\sin x}{(-1-1)}-\frac{1}{2}\frac{1}{(-1-1)}\cos x+\frac{1}{4}e^{-x}=\frac{1}{2}-\frac{\sin x}{4}+\frac{\cos x}{4}+\frac{1}{4}e^{-x}$

$$\text{P.I.}=\frac{1}{2}+\frac{1}{4}(\cos x-\sin x+e^{-x})$$

Hence, the complete solution is

$$y=C_1e^{-x}+e^{\frac{x}{2}}\left[C_2\cos\frac{\sqrt{3}}{2}x+C_3\sin\frac{\sqrt{3}}{2}x\right]+\frac{1}{2}+\frac{1}{4}(\cos x-\sin x+e^{-x})$$ **Ans.**

5.3 Summary

Within a particular class of ordinary differential equations (ODEs), known as homogeneous differential equations, every term may be represented as a function of the dependent variable and its derivatives. When modeling systems where all terms may be represented as homogeneous functions of the dependent variable and its derivatives, homogeneous differential equations offer a useful foundation. Their answers provide understanding of these systems' behavior and are crucial resources for mathematical modeling and research.

5.4 Keywords

- Differential equation
- Homogeneous differential equation
- Complementry function
- Particular inegral

5.5 Self Assessment Questions

1. $\frac{d^2y}{dx^2}+6y=\sin 4x$ **Ans.** $C_1\cos\sqrt{6}x+C_2\sin\sqrt{6}x-\frac{1}{10}\sin 4x$
2. $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+3x=\sin t$ **Ans.** $e^{-t}[A\cos\sqrt{2}t+B\sin\sqrt{2}t]-\frac{1}{4}(\cos t-\sin t)$
3. $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=\sin 2t$, given that when $t=0$, $x=3$ and $\frac{dx}{dt}=0$

   Ans. $e^{-t}\left[\frac{55}{17}\cos 2t+\frac{53}{34}\sin 2t\right]-\frac{1}{17}(4\cos 2t-\sin 2t)$

5.6 Case Study

An analysis of electrical circuits:

When conventional techniques become unfeasible owing to circuit complexity or nonlinearity, series solutions can be utilized in electrical engineering to examine intricate circuits.

Question: Learn about the behavior and functionality of a complicated electrical circuit by analyzing it.

5.7 References

1. "Elementary Differential Equations and Boundary Value Problems" by William E. Boyce and Richard C. DiPrima
2. "Advanced Engineering Mathematics" by Erwin Kreyszig

Unit-6

Total Differential Equations

Learning Objectives:

- Develop an understanding of fundamental geometric concepts such as points, lines, planes, curves, surfaces, and their properties.
- Enhance their ability to visualize geometric objects and transformations in two-dimensional and three-dimensional space.
- Investigate geometric properties such as distance, angle, area, volume, curvature, and symmetry, and understand their significance in various contexts.

Structure:

6.1 Total Differential Equations

For function z = f(x, y) whose partial derivatives exists, total differential of z is

$$dz = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy,$$

One can generalize total differentials. When considering a function f = f(x, y, z) with partial derivatives, the total differential of *f* can be found using

$$df = f_x(x, y, z) \cdot dx + f_y(x, y, z) \cdot dy + f_z(x, y, z) \cdot dz.$$

Ex 1: Find the differential equation corresponding to the surfacexy = c(a-z) where c is a parameter.

Solution

$$\frac{xy}{a-z} = c$$

The total derivative of the above relation gives

$$d\left(\frac{xy}{a-z}\right) = 0$$

$$\Rightarrow \frac{(a-z)\,d(xy) - xy\,d(a-z)}{(a-z)^2} = 0$$

$$\Rightarrow (a-z)\,[x\,dy + y\,dx] + xy\,dz = 0$$

which is the required differential equation corresponding to the given family of surfaces.

Ex 2: Find the differential equation corresponding to the family of surfaces $x^2 + y^2 + z^2 = xc$ where c is a parameter.

Solution

$$\frac{x^2 + y^2 + z^2}{x} = c$$

The total derivative of the above equation gives

$$d\left(\frac{x^2 + y^2 + z^2}{x}\right) = 0$$

$$\Rightarrow d\left(\frac{x^2}{x}\right) + d\left(\frac{y^2}{x}\right) + d\left(\frac{z^2}{x}\right) = 0$$

$$\Rightarrow dx + \frac{2xydy - y^2dx}{x^2} + \frac{2xzdz - z^2dx}{x^2} = 0$$

$$\Rightarrow x^2dx + 2xydy - y^2dx + 2xzdz - x^2dx = 0$$

$$\Rightarrow (x^2 - y^2 - z^2)dx + 2xydy + 2xzdz = 0$$

as the required total differential equation.

6.2 Solutions and Conditions

Let us consider the following differential equations

$$3x^2(y+z)dx+(z^2+x^3)dy+(2yz+x^3)dz=0 \qquad (7)$$

$$(3xz+2y)dx+xdy+x^2dz=0 \qquad (8)$$

$$ydx+(z-y)dy+xdz=0 \qquad (9)$$

They are all being of the form of Eqn. (1) are total differential equations.

Eqn. (7) is an exact differential of the function

$$f(x,\ y,\ z)=x^3y+x^3z+z^2y=c$$

where c is an arbitrary constant.
You can easily check that

$$d[x^3y+x^3z+z^3y]=0$$

$$\Rightarrow 3x^2ydx+x^3dy+3x^2zdx+x^3dz+z^2dy+2zydz=0$$

$$\Rightarrow 3x^2(y+z)dx+(x^3+z^2)dy+(x^3+2zy)dz=0$$

which is our Eqn. (7). Such an equation is called an **exact equation**. Thus Eqn. (7) is an exact equation.
Eqn. (8) is not an exact differential, but the use of x as an integrating factor yields

$$(3x^2z+2xy)dx+x^2dy+x^3dz=0$$

which is the exact differential of the function

$$f(x,\ y,z)=x^3z+x^2y=c,\ c \text{ being a constant.}$$

Eqn. (7) and (8) are called **integrable equations**.
Further, you can see that Eqn. (9) is not integrable as no function

6.3 Geometrical Interpretation and Examples

A partial derivative has the same geometric interpretation as an ordinary derivative. It shows the tangent's slope to the curve that the function at a certain point P represents. In the instance of a two-variable function

Z=(x, y)

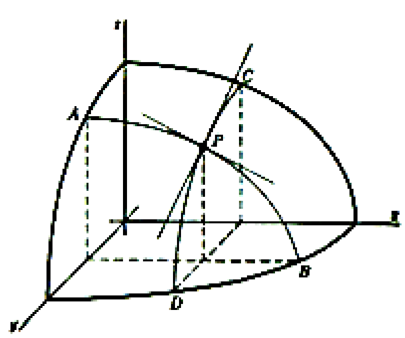

Figure 6.1: Geometrical Interpretation partial derivative

Fig. 6.1 shows the interpretation of$\frac{\partial f}{\partial x}$ and of$\frac{\partial f}{\partial y}$. $\frac{\partial f}{\partial x}$ corresponds to the slope of the tangent to the curve APB at point P. Similarly, $\frac{\partial f}{\partial y}$corresponds to the slope of the tangent to the curve CPD at point P

6.4 Summary

Geometrical interpretation involves understanding fundamental geometric concepts such as points, lines, curves, and surfaces, and their properties in two-dimensional and three-dimensional space. Through visualization and analytical reasoning, students explore transformations, coordinate systems, and geometric properties like distance, angle, and curvature. Geometrical interpretation extends to diverse applications in science, engineering, and art, where geometric modeling, analysis, and visualization play crucial roles. By mastering geometrical interpretation, students develop problem-solving skills, critical thinking abilities, and an appreciation for the beauty and utility of geometry across various disciplines.

6.5 Keywords

- Geometrical Interpretation
- Curvature

6.6Self-Assessment Questions

1. How does understanding geometric properties aid in solving real-world problems?
2. Can you explain the significance of coordinate systems in geometrical interpretation?
3. What are some common geometric transformations, and how do they affect geometric objects?
4. How does curvature influence the shape and behavior of curves and surfaces?
5. In what ways do geometric concepts intersect with other disciplines, such as physics or computer science?
6. How does visualization enhance our understanding of geometric relationships and structures?
7. What role does symmetry play in geometric interpretation and analysis?

8. How do we use geometric reasoning to prove theorems and solve geometric puzzles?
9. Can you provide examples of how geometric interpretation is applied in engineering or architecture?
10. What historical developments have shaped our understanding of geometry, and how do they influence modern applications?

6.7 Case Study

Geometric interpretation plays a fundamental role in computer graphics, where visual representations of objects and scenes are created and manipulated using mathematical models. From rendering lifelike images to simulating virtual environments, geometric interpretation enables the creation of immersive visual experiences in various applications, including gaming, animation, virtual reality, and computer-aided design (CAD).

Objective: To explore how geometric interpretation is applied in computer graphics and its significance in creating realistic and interactive digital environments.

6.8 References

1. Foley, J. D., van Dam, A., Feiner, S. K., & Hughes, J. F. (2020). Computer Graphics: Principles and Practice. Addison-Wesley.
2. Rogers, D. F., & Adams, J. A. (2019). Mathematical Elements for Computer Graphics (2nd ed.). McGraw-Hill.

Unit-7

Second Order Ordinary Differential Equations with Variable Coefficients

Learning Objectives:

- To understand Differential equations with Variable Coefficients
- To understand Equations of first order and first degree

Structure

7.1 Introduction:-

In ordinary differential equations (ODEs), variables typically represent quantities that change with respect to one or more independent variables. The most common independent variable is denoted by tt and often represents time, but it can also represent other quantities like spatial position or another independent parameter.

Here are some common variables and their meanings in ODEs:

1. Dependent variable: Denoted by y or another letter, it represents the quantity that we're trying to solve for. This quantity depends on the independent variable(s) and possibly its derivatives.
2. Independent variable: Denoted by t or another letter, it represents the variable with respect to which the dependent variable and its derivatives are defined. For example, in many physical problems, $t$represents time.
3. Parameters: Parameters are constants or fixed quantities that appear in the ODE but do not vary with respect to the independent variable. They often represent physical constants or initial/boundary conditions.
4. Functions of independent variables: These are additional functions that may appear in the ODE, either on the right-hand side or as coefficients. They can be functions of the independent variable(s) or constants.
5. Derivatives: Derivatives of the dependent variable with respect to the independent variable(s) often appear in ODEs. They represent rates of change or slopes of the dependent variable.

- k is a parameter, representing a constant rate of change.
- $dydt$ is the derivative of yy with respect to tt, representing the rate of change of y with respect to time.

7.2 Equation whose one solution is known:

If $y = u$ is given solution belonging to the complementary function of the differential equation. Let the other solution be $y = v$. Then $y = u.\ v$ is complete solution of the differential equation.

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ....(1), be the differential equation and u is the solution included in the complementary function of (1)

$$\therefore \qquad \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad ...(2)$$

$$y = u.\ v$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + u\frac{d^2v}{dx^2}$$

Substituting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1), we get

$$v\frac{d^2u}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + u\frac{d^2v}{dx^2} + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Qu.v = R$$

On arranging

$$\Rightarrow \ v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] + u\left[\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right] + 2\frac{du}{dx}\cdot\frac{dv}{dx} = R$$

The first bracket is zero by virtue of relation (2), and the remaing is divided by u.

$$\frac{d^2v}{dx^2} + P\frac{dv}{dx} + \frac{2}{u}\frac{du}{dx}\frac{dv}{dx} = \frac{R}{u}$$

$$\Rightarrow \qquad \frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u} \qquad ...(3)$$

Let $\qquad \frac{dv}{dx} = z$, so that $\frac{d^2v}{dx^2} = \frac{dz}{dx}$

Equation (3) becomes $\boxed{\frac{dz}{dx} + \left[P + \frac{2}{u}\frac{du}{dx}\right]z = \frac{R}{u}}$

This is the linear differential equation of first order and can be solved (z can be found), which will contain one constant.

On integration $z = \frac{dv}{dx}$, we can get v.

Having found v, the solution is $y = uv$.

Note: Rule to find out the integral belonging to the complementary function

Rule	*Condition*	u
1	$1 + P + Q \quad = 0$	e^x
2	$1 - P + Q \quad = 0$	e^{-x}
3	$1 + \frac{P}{a} + \frac{Q}{a^2} \quad = 0$	e^{ax}
4	$P + Qx \quad = 0$	x
5	$2 + 2Px + Qx^2 \quad = 0$	x^2
6	$n(n-1) + Pnx + Qx^2 \quad = 0$	x^n

Example 1: Solve

$y'' - 4xy' + (4x^2 - 2)y = 0$ given that $y = e^{x^2}$ is an integral included in the complementary function.

Solution. $y'' - 4xy' + (4x^2 - 2)y = 0$ ...(1)

On putting $y = v.e^{x^2}$ in (1), the reduced equation as in the article 3.36

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = 0 \qquad [P = -4x,\ Q = 4x^2 - 2,\ R = 0]$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-4x + \frac{2}{e^{x^2}}(2x\,e^{x^2})\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + [-4x + 4x]\frac{dv}{dx} = 0 \qquad \Rightarrow \qquad \frac{d^2v}{dx^2} = 0 \Rightarrow \frac{dv}{dx} = c, \Rightarrow v = c_1x + c_2$$

$$\therefore \qquad y = uv \qquad [u = e^{x^2}]$$

$$y = e^{x^2}(c_1x + c_2)$$ **Ans.**

Example 2:

Solve $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = 0$

given that $y = e^x$ is an integral included in the complementary function.

Solution. $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)\,y = 0$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{2x-1}{x}\frac{dy}{dx} + \frac{x-1}{x}y = 0 \qquad [1 + P + Q = 0] \qquad ...(1)$$

By putting $y = ve^x$ in (1),we get the reduced equation as in the article 3.36.

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = 0 \qquad ...(2)$$

Putting $u = e^x$ and $\dfrac{dv}{dx} = z$ in (2), we get $\dfrac{dz}{dx} + \left[-\dfrac{2x-1}{x} + \dfrac{2}{e^x}e^x\right]z = 0$

$$\Rightarrow \frac{dz}{dx} + \frac{-2x+1+2x}{x}z = 0 \qquad \Rightarrow \qquad \frac{dz}{dx} + \frac{z}{x} = 0$$

$$\Rightarrow \qquad \frac{dz}{z} = -\frac{dx}{x} \Rightarrow \log z = -\log x + \log c_1$$

$$\Rightarrow \qquad z = \frac{c_1}{x} \text{ or } \frac{dv}{dx} = \frac{c_1}{x} \text{ or } dv = c_1\frac{dx}{x} \Rightarrow v = c_1 \log x + c_2$$

$$y = u.\,v = e^x(c_1 \log x + c_2)$$ **Ans.**

Example 3:

Solve $x^2 \dfrac{d^2y}{dx^2} - 2x[1+x]\dfrac{dy}{dx} + 2(1+x)y = x^3$

Solution. $x^2 \dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)y = x^3$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{2x(1+x)}{x^2}\frac{dy}{dx} + \frac{2(1+x)y}{x^2} = x \qquad ...(1)$$

Here $\qquad P + Qx = -\dfrac{2x(1+x)}{x^2} + \dfrac{2(1+x)}{x^2}x = 0$

Hence $y = x$ is a solution of the C.F. and the other solution is v.

Putting $y = vx$ in (1), we get the reduced equation as in article 3.36

$$\frac{d^2v}{dx^2} + \left\{P + \frac{2}{u}\frac{du}{dx}\right\}\frac{du}{dx} = \frac{x}{u}$$

$$\frac{d^2v}{dx^2} + \left[\frac{-2x(1+x)}{x^2} + \frac{2}{x}(1)\right]\frac{dv}{dx} = \frac{x}{x}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \Rightarrow \frac{dz}{dx} - 2z = 1 \qquad \left[\frac{dv}{dx} = z\right]$$

which is a linear differential equation of first order and $I.F. = e^{\int -2\,dx = e^{-2x}}$

Its solution is $z\,e^{-2x} = \int e^{-2x}\,dx + c_1$

$$z\,e^{-2x} = \frac{e^{-2x}}{-2} + c_1 \text{ or } z = \frac{-1}{2} + c_1 e^{2x}$$

$$\Rightarrow \quad \frac{dv}{dx} = -\frac{1}{2} + c_1 e^{2x} \text{ or } dv = \left(-\frac{1}{2} + c_1 e^{2x}\right)dx \Rightarrow v = \frac{-x}{2} + \frac{c_1}{2}e^{2x} + c_2$$

$$y = uv = x\left(\frac{-x}{2} + \frac{c_1}{2}e^{2x} + c_2\right) \qquad \textbf{Ans.}$$

Exercise

Solve

1. $(3-x)\dfrac{d^2y}{dx^2} - (9-4x)\dfrac{dy}{dx} + (6-3x)y = 0$, given $y = e^x$ is a solution.

Ans. $y = \dfrac{c_1}{8}e^{3x}(4x^3 - 42x^2 + 150x - 183) + c_2 e^x$

2. $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + (1-x)y = x^2 e^{-x}$ given $y = e^x$ is an integral included in C.F.

Ans. $y = c_2 e^x + c_1(2x+1)e^{-x} - \dfrac{1}{4}(2x^2 + 2x + 1)e^{-x}$

3. $(1-x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x(1-x^2)^{3/2}$, given $y = x$ is part of C.F.

Ans. $y = -\dfrac{x}{9}(1-x^2)^{3/2} - c_1[\sqrt{(1-x^2)} + x\sin^{-1}x] + c_2 x.$

7.3 Normal Form:-

Consider the differential equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ ...(1)

Put $y = uv$ where v is not an integral solution of C.F.

$$\frac{dy}{dx} = v\frac{du}{dx}+u\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}$$

On putting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1) we get

$$\left(u\frac{d^2v}{dx^2}+2\frac{dv}{dx}\frac{du}{dx}+v\frac{d^2u}{dx^2}\right)+P\left(u\frac{du}{dx}+v\frac{du}{dx}\right)+Q.uv=R$$

$$\Rightarrow \quad v\frac{d^2u}{dx^2}+\frac{du}{dx}\left(Pv+2\frac{dv}{dx}\right)+u\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Q.v\right)=R$$

$$\Rightarrow \quad \frac{d^2u}{dx^2}+\frac{du}{dx}\left(P+\frac{2}{u}\frac{dv}{dx}\right)+\frac{u}{v}\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Q.v\right) = \frac{R}{v} \qquad ...(2)$$

Here in the last bracket on L.H.S. is not zero $y = v$ is not a part of C.F.
Here we shall remove the first derivative.

$$P+\frac{2}{v}\frac{dv}{dx}=0 \text{ or } \frac{dv}{v}=-\frac{1}{2}P\,dx \text{ or } \log v=\frac{-1}{2}\int P\,dx$$

$$v = e^{-\frac{1}{2}\int P\,dx}$$

In (2) we have to find out the value of the last bracket *i.e.*, $\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv$

$$\frac{dv}{dx} = -\frac{P}{2}e^{-\frac{1}{2}\int P\,dx} = -\frac{1}{2}Pv \qquad \left[\therefore v=e^{-1/2\int pdx}\right]$$

$$\frac{d^2v}{dx^2} = -\frac{1}{2}\frac{dP}{dx}v-\frac{P}{2}\frac{dv}{dx}=-\frac{1}{2}\frac{dP}{dx}v-\frac{P}{2}\left(-\frac{1}{2}Pv\right)=-\frac{1}{2}\frac{dP}{dx}v+\frac{1}{4}P^2v$$

$$\therefore \frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv = -\frac{1}{2}\frac{dP}{dx}v+\frac{1}{4}P^2v+P\left(-\frac{1}{2}Pv\right)+Qv = v\left[Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2\right]$$

Equation (1) is transformed as

$$\frac{d^2u}{dx^2}+\frac{u}{v}v\left\{Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}\right\}=\frac{R}{v}$$

$$\Rightarrow \quad \frac{d^2u}{dx^2}+u\left\{Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}\right\} = Re^{\frac{1}{2}\int P\,dx}$$

$$\frac{d^2u}{dx^2}+Q_1u = R_1 \qquad \text{where } Q_1=\left[Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}\right], \quad R_1=Re^{\frac{1}{2}\int P\,dx} \text{ or } \frac{R}{v}$$

$$y = uv \qquad \text{and} \qquad v=e^{-\frac{1}{2}\int P\,dx}$$ **Ans.**

Example 4:

Solve $\frac{d}{dx}\left[\cos^2 x\frac{dy}{dx}\right]+\cos^2 x.\, y=0$

Solution. We have, $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right)+\cos^2 x.\, y=0$

$\Rightarrow \quad \frac{d^2y}{dx^2}\cos^2 x-2\cos x\sin x\frac{dy}{dx}+(\cos^2 x)y=0 \;\Rightarrow\; \frac{d^2y}{dx^2}-2\tan x.\frac{dy}{dx}+y=0$

Here, $\quad P=-2\tan x,\ Q=1,\ R=0$

$$Q_1=Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}=1-\frac{1}{2}(-2\sec^2 x)-\frac{4\tan^2 x}{4}$$

$$=1+\sec^2x-\tan^2x=1+1=2$$

$$R_1=Re^{\frac{1}{2}\int P\,dx}=0$$

$$v=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}\int(-2\tan x)dx}=e^{\int\tan x\,dx}=e^{\log\sec x}=\sec x$$

Normal equation is

$$\frac{d^2u}{dx^2}+Q_1u=R_1$$

$$\frac{d^2u}{dx^2}+2u=0 \quad \text{or} \quad (D_2+2)\,u=0 \quad \Rightarrow \quad D=\pm i\sqrt{2}$$

$$u=c_1\cos\sqrt{2}\,x+c_2\sin\sqrt{2}\,x$$

$$y=u.v$$

$$=[c_1\cos\sqrt{2}\,x+c_2\sin\sqrt{2}x]\sec x$$ **Ans.**

Example 6:

Solve $x^2\frac{d^2y}{dx^2}-2(x^2+x)\frac{dy}{dx}+(x^2+2x+2)y=0$

Solution. We have, $\frac{d^2y}{dx^2}-\frac{2(x^2+x)}{x^2}\frac{dy}{dx}+\left(\frac{x^2+2x+2}{x^2}\right)y=0$ ...(1)

Here $p=-2\left(1+\frac{1}{x}\right), Q=\frac{x^2+2x+2}{x^2},\ R=0$

In order to remove the first derivative, we put $y=u.v$ in (1) to get the normal equation

$$\frac{d^2v}{dx^2}+Q_1v=R_1 \qquad ...(2)$$

where $v = e^{-\frac{1}{2}\int p\,dx} = e^{-\frac{1}{2}\int -2\left(1+\frac{1}{x}\right)dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x}\,.\,e^{\log x} = x\,e^{x}$

$Q_1 = Q - \frac{1}{2}\frac{dp}{dx} - \frac{p^2}{4} = \frac{x^2+2x+2}{x^2} - \frac{1}{2}\left(\frac{2}{x^2}\right) - \frac{4}{4}\left(1+\frac{1}{x}\right)^2 = 1 + \frac{2}{x} + \frac{2}{x^2} - \frac{1}{x^2} - 1 - \frac{1}{x^2} - \frac{2}{x}$

$R_1 = R\,e^{\frac{1}{2}\int p\,dx} = 0$

On putting the values of Q_1 and R_1 in (2), we get $\frac{d^2u}{dx^2} + 0(u) = 0 \Rightarrow \frac{d^2u}{dx^2} = 0$

$$\frac{du}{dx} = c_1 \Rightarrow u = c_1 x + c_2$$

$\therefore$ $\quad y = u.v = (c_1 x + c_2)\,x\,e^{x}$ **Ans.**

Exercise

Solve

1. $\frac{d^2y}{dx^2} - 2\tan x.\,y - 5y = 0$ **Ans.** $y = (a\,e^{2x} + e^{-3x})\sec x$
2. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-3)\,y = e^{x^2}$ **Ans.** $y = (c_1 e^{x} + c_2\,e^{-x} - 1)$
3. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = e^{\frac{1}{2}(x^2+2x)}$ **Ans.** $y = (c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}\,x)\,e^{\frac{x^2}{2}+\frac{1}{4}e^{x}.\,e^{\frac{x^2}{2}}}$
4. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(n^2 + \frac{2}{x^2}\right)y = 0$ **Ans.** $y = (c_1\cos nx + c_2\sin nx)x$
5. $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - n^2 y = 0$ **Ans.** $y = (c_1 e^{nx} + c_2 + e^{-nx})\frac{1}{x}$

7.4 Change of Independent Variable

Consider, $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ...(1)

Let us change the independent variable x to z and $z = f(x)$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \quad\Rightarrow\quad \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\left(\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{dy}{dz}\frac{d^2z}{dx^2}\right)+P\left(\frac{dy}{dz}\frac{d^2z}{dx^2}\right)+Qy=R$$

$$\Rightarrow \quad \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\left(P\frac{dz}{dx}+\frac{d^2z}{dx^2}\right)\frac{dy}{dz}+Qy=R$$

$$\Rightarrow \quad \frac{d^2y}{dz^2}+\frac{P\left(\frac{dz}{dx}+\frac{d^2z}{dx^2}\right)}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz}+\frac{Qy}{\left(\frac{dz}{dx}\right)^2}=\frac{R}{\left(\frac{dz}{dx}\right)^2} \Rightarrow \frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1 \qquad ...(2)$$

where $P_1=\dfrac{P\left(\frac{dz}{dx}+\frac{d^2z}{dx^2}\right)}{\left(\frac{dz}{dx}\right)^2}$, $\quad Q_1=\dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$, $\quad R_1=\dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

Equation (2) is solved either by taking $P_1=0$ or $Q_1=a$ constant.

Equation (2) can be solved by by two methods, by taking

First Method, $\quad P_1=0$

Second Method, $\quad Q=$ constant

Working Rule

Step 1. Coefficient of $\frac{d^2y}{dx^2}$ should be made as 1 if it is not so.

Step 2. To get P, Q and R, compare the given differential equation with the standard form $y''+Py'+Qy=R$.

Step 3. Find P_1, Q_1 and R_1 by the following formulae.

$$P_1=\frac{\frac{d^2y}{dx^2}+p\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2},\quad R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Step 4. Find out the value of z by taking

First Method, $P_1=0$ $\qquad$ **Second Method.** $Q_1=$ constant

Step 5. We get a reduced equation $\dfrac{d^2y}{dz^2}+P_1\dfrac{dy}{dz}+Q_1y=R_1$

On solving this equation we can find out the value of y in terms of z.

Then write down the solution in terms of x by replacing the value of z.

Example 7:

Solve $\dfrac{d^2y}{dx^2}+\cot x\dfrac{dy}{dx}+4y\,\text{cosec}^2 x=0$

Solution. We have, $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4\,y \operatorname{cosec}^2 x = 0$ ...(1)

Here, $P = \cot x$, $Q = 4 \operatorname{cosec}^2 x$ and $R = 0$

Changing the independent variable from x to z, the equation becomes

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dz} + Q_1\, y = 0 \qquad ...(2)$$

where
$$P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

Case I. Let us take $P_1 = 0$

$$\frac{p\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} = 0 \text{ or } P\frac{dz}{dx} + \frac{d^2z}{dx^2} = 0 \Rightarrow \frac{d^2z}{dx^2} + \cot x \frac{dz}{dx} = 0 \qquad ...(3)$$

Put
$$\frac{dz}{dx} = v, \quad \frac{d^2z}{dx^2} = \frac{dv}{dx}$$

(3) becomes $\frac{dv}{dx} + (\cot x)\, v = 0 \Rightarrow \frac{dv}{v} = -\cot x.\, dx$

$\Rightarrow$ $\log v = -\log \sin x + \log c = \log c \log c \operatorname{cosec} x \Rightarrow v = c \operatorname{cosec} x$

$$\frac{dz}{dx} = c \operatorname{cosec} x \Rightarrow dz = (c \operatorname{cosec} x)\, dx \Rightarrow z = c \log \tan \frac{x}{2}$$

Case II.

Now let us take Q_1 = Constant.

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4 \operatorname{cosec}^2 x}{c^2 \operatorname{cosec}^2 x} = \frac{4}{c^2} \text{ which is constant}$$

Hence the equation (2) reduces to

$$\frac{d^2y}{dz^2} + 0\frac{dy}{dz} + \frac{4}{c^2}y = 0 \text{ or } \frac{d^2y}{dz^2} + \frac{4}{c^2}y = 0 \qquad \left[\because P_1 = 0, Q_1 = \frac{4}{c^2}\right]$$

$$\Rightarrow \left(D^2 + \frac{4}{c^2}\right)y = 0, \qquad \text{A.E. is} \quad m^2 + \frac{4}{c^2} = 0 \Rightarrow m = \pm i\,\frac{2}{c}$$

$$\text{C.F.} = c_1 \cos\frac{2z}{c} + c_2 \sin\frac{2z}{c} \qquad \left(z = c \log \tan \frac{x}{2}\right)$$

$$\Rightarrow \quad y = c_1 \cos\left(2 \log \tan\frac{x}{2}\right) + c_2 \sin\left(2 \log \tan\frac{x}{2}\right) \qquad \textbf{Ans.}$$

Example 8:

Solve $x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2 y = \frac{1}{x^2}$

Solution. We have, $\frac{d^2y}{dx^2}+\frac{3}{x}\frac{dy}{dx}+a^2\frac{y}{x^6}=\frac{1}{x^8}$ ...(1)

Here $P=\frac{3}{x}$ and $Q=\frac{a^2}{x^6}$

On changing the independent variable x to z, the equation (1) is reduced to

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1\, y=R_1 \qquad ...(2)$$

Using Second Method

Let $Q_1=a_2$ (constant) $\qquad Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{a^2}{x^6\left(\frac{dz}{dx}\right)^2}=$ constant $=a^2$ (say)

$$\therefore \quad x^6\left(\frac{dz}{dx}\right)^2=1 \Rightarrow x^3\left(\frac{dz}{dx}\right)=1 \Rightarrow \frac{dz}{dx}=\frac{1}{x^3} \Rightarrow dz=\frac{dx}{x^3} \Rightarrow z=\frac{x^{-2}}{-2}+c$$

On differentiating twice, we have $\frac{d^2z}{dx^2}=\frac{-3}{x^4}$

$$P_1=\frac{P\frac{dz}{dx}+\frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}=\frac{\frac{3}{x}\cdot\frac{1}{x^3}+\left(\frac{-3}{x^4}\right)}{\left(\frac{1}{x^3}\right)^2}=0 \Rightarrow R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}=\frac{\frac{1}{x^8}}{\frac{1}{x^6}}=\frac{1}{x^2}=-2z$$

On putting the values of P_1, Q_1 and R_1 in (2), we get

$$\frac{d^2y}{dz^2}+a^2y=-2z \qquad \Rightarrow \qquad (D^2+a^2)\,y=-2\,z$$

A.E. is $m^2+a^2=0$, $\quad m=\pm\, i\, a$, $\qquad \Rightarrow \qquad$ C.F. $=c_1\cos az+c_2\sin az$

$$\text{P.I.}=\frac{1}{D^2+a^2}(-2z)=\frac{1}{a^2}\left[1+\frac{D^2}{a^2}\right]^{-1}(-2z)=\frac{1}{a^2}\left[1-\frac{D^2}{a^2}\right](-2z)=\frac{-2z}{a^2}=\frac{1}{a^2x^2}$$

$$y=\text{C.F.}+\text{P.I.}$$

$$y=c_1\cos\frac{a}{2x^2}-c_2\sin\frac{a}{2x^2}+\frac{1}{a^2x^2}$$ **Ans.**

7.5 Summary

By dissecting the issue into more manageable ordinary differential equations (ODEs), separation of variables is a potent strategy for solving partial differential equations (PDEs). The essential concept is to presume that the PDE's solution may be written as the product of functions, each of which depends only on one variable.

7.6 Keywords

- Normal Form
- Independent Variable
- Dependent variable
- Independent variable

7.7 Self Assessment Questions

1. $x^4\frac{d^2y}{dx^2}+2x^3\frac{dy}{dx}+a^2y=0$ **Ans.** $y=c_1\cos\frac{a}{x}+c_2\sin\frac{a}{x}$
2. $\cos x\frac{d^2y}{dx^2}+\frac{dy}{dx}\sin x-2y\cos^3 x=2\cos^5 x$ **Ans.** $y=c_1e^{\sqrt{2}\sin x}+c_2e^{-\sqrt{2}\sin x}+\sin^2 x$
3. $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0$ **Ans.** $y=c_1\cos(\sin x)+c_2\sin(\sin x)$
4. $x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=8x^2\sin x^2$ **Ans.** $y=c_1e^{x^2}+c_2e^{\cos x}+\frac{1}{6}e^{-\cos x}$
5. $\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2 x=e^{-\cos x}\sin^2 x$ **Ans.** $y=c_1e^{2\cos x}+c_2e^{\cos x}+\frac{1}{6}e^{-\cos x}$
6. $\frac{d^2y}{dx^2}+(\tan x-1)^2\frac{dy}{dx}-n(n-1)y\sec^4 x=0$ **Ans.** $y=C_1e^{-n\tan x}+C_2e^{(n-1)\tan x}$
7. $\frac{d^2y}{dx^2}-\cot x\frac{dy}{dx}-y\sin^2 x=\cos x-\cos^3 x$ **Ans.** $y=C_1e^{-\cos x}+C_2e^{\cos x}-\cos x$
8. $\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2 x=e^{-\cos x}\sin^2 x$ **Ans.** $y=C_1e^{\cos x}+C_2e^{2\cos x}+\frac{1}{6}e^{-\cos x}$

7.8 Case Study

A non-mathematical context: designing a multi-stage distillation column for separating components in a chemical process.

Problem Statement: Imagine a chemical engineering company tasked with designing a distillation column to separate a mixture of ethanol and water into its pure components. The goal is to achieve high purity ethanol as the top product and high purity water as the bottom product.

7.9 References

1. "Elementary Differential Equations and Boundary Value Problems" by William E. Boyce and Richard C. DiPrima
2. "Advanced Engineering Mathematics" by Erwin Kreyszig

Unit-8

Partial Differential Equations (PDEs)

Learning Objectives:

- Define partial differential equations (PDEs) of second order and distinguish them from other types of PDEs, such as first-order PDEs.
- Identify and derive the canonical forms of second-order PDEs, such as the Laplace, heat, and wave equations, in various coordinate systems.
- Discuss numerical techniques, such as finite difference, finite element, and spectral methods, for approximating solutions to PDEs.

Structure

8.1 Partial differential Equations:-

An equation with z as the dependent variable and x, y, and z as the independent variables, such that z = f(x,y), is called a partial differential equation. We also use the notations

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s,\ \frac{\partial^2 z}{\partial y^2} = t.$$

8.2 Order of a Partial differential Equations

A partial differential equation's order is determined by the highest partial derivative that appears in the equation, and its degree is determined by that derivative's degree.

For example, (1) x + y

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Here, z is dependent and x, y are independent and this equation is of order one and degree one.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 z}{\partial^2 y} + \frac{\partial^2 v}{\partial z^2} = 0$$

This is an order two, degree equation where v is dependent and x, y, and z are independent. The study of wave equations, heat equations, electromagnetic, radar, ratios, television, and other subjects will all heavily rely on partial differential equations.

Formation of Partial Differential Equation(PDE)

PDE can be obtained by:

(i) Elimination of arbitrary constants

(ii) Elimination of arbitrary functions involving two or more variables.

(i) Elimination of Arbitrary Constants

Let f(x, y, z,a,b) = 0 ...(1)

consist of an equation with the two arbitrary constants "a" and "b." Partially differentiating this equation in relation to x and y yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial x}\right) = 0 \qquad \ldots(2)$$

8.3 Lagrange's Method and Standard Forms

An equation of the form Pp + Qq = R is said to be Lagrange's type of partial differential equations.

Steps for solving Pp+Qq=R by Lagrange's method

STEP 1: Insert the first-order linear partial differential equation that has been provided in the standard from

Pp+Qg=R ………..(1)

STEP 2:Note down the following Lagrange's auxiliary equation for (1):

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{..............(2)}$$

STEP 3:Apply the established techniques to solve (2). As two independent solutions of (2), let u(x, y, z) = c1 and v(x, y, z) = c_2.

STEP 4: The general solution (or integral) of (1) is then written in one of the following three equivalent forms:$\varphi(u, v) = 0$, $u = \varphi(v) or, v = \varphi(u), \varphi$being an arbitrary function.

Example 1:Solve a(p+q) = z

Solution: Given $ap + aq = z$

The Lagrange's Auxiliary equation

$$\frac{dx}{a} = \frac{dy}{a} = \frac{dz}{1}$$

$$dx - dy = 0$$

Integrating

$x - y = c_1$

Taking the last two members

$$dy - a\,dz = 0$$

Integrating

$$y - az = c2$$

the required solution is given by

ϕ(x- y, y – az) = 0, ϕ being an arbitrary function.

Example 2: Solve

$(mz - ny)p + (nx - lz)q = ly - mx$

Sol. The Lagrange's auxiliary equation of the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Changing x, y, z as multipliers, each fraction

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

Therefore $\quad xdx + ydy + zdz = 0$

or $\quad 2xdx + 2ydy + 2zdz = 0$

Integrating, $\quad x^2 + y^2 + z^2 = c_1$.

$$= \frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)}$$

$$= \frac{ldx + mdy + n\,dz}{0}$$

Therefore $\quad ldx + mdy + n\,dz = 0$

so that $\quad lx + my + nz = c_2$

the required general solution is given by

$$\phi(x^2 + y^2 + z^2,\ ldx + mdy + n\,dz) = 0$$

Example 3: Solve

$$\left(\frac{y^2z}{x}\right)p + zxq = y^2$$

Solution: The Lagrange's auxiliary equations are

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{zx} = \frac{dz}{y^2}$$

or $$\frac{xdx}{y^2z} = \frac{dy}{zx} = \frac{dz}{y^2}$$

From 1st and 2nd fractions we get

$$\frac{xdx}{y^2z} = \frac{dy}{zx}$$

or $$x^2dx = y^2dy$$

integrating

$$\frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3}$$

$$c_1 = x^3 - y^3$$

From 1st and 3rd fractions we get

$$\frac{xdx}{y^2z} = \frac{dz}{y^2}$$

or $$xdx = zdz$$

integrating

or $$\frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

or $$c_2 = x^2 - z^2$$

The general solution is given by

$$f(c_1, c_2) = 0$$

$$f(x^3 - y^3, x^2 - z^2) = 0$$

8.4Charpit's Method

In the event that the provided equation cannot be reduced to one of the four first-order non-linear partial differential equation types mentioned above, we solve all first-order partial differential equations using a method developed by Charpit. This method is known as Charpit's method.

$$F(x, y, z, p, q) = 0$$

Since z depend on x and y, we have,

$$dZ = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = Pdx + Qdy$$

Now, if we can find another relation between x, y, z, p, q such that,

$$f(x, y, z, p, q) = 0$$

Example4 :

Solve $(p^2 + q^2)y = qz$.

Solution: Let $F(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ ...(1)

The subsidiary equations are:

$$\frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

The last two fractions yield $pdp + qdq = 0$

which on integration gives

$p^2 + q^2 = c^2$ ...(2)

In order to solve equations (1) and (2), put $p^2 + q^2 = c^2$ in equation (1) so that $q = \frac{c^2y}{z}$

Now, substituting this value of q in equation (2), we get

$$p = c\sqrt{\frac{z^2-c^2y^2}{z}}$$

Hence, $dz = pdx + qdy = \frac{c}{2}\sqrt{(z^2 - c^2y^2)dx} + \frac{c^2y}{z}dy$

$=> zdz - c^2ydy = c\sqrt{z^2 - c^2y^2}dx$

$=> \frac{(1/_2)d(z^2-c^2y^2)}{\sqrt{z^2-c^2y^2}} = cdx$

Integrating, we get the required solution as $z^2 = (a + cx)^2 + c^2y^2$

Example 5:

Solve $2zx - px^2 - 2qxy + pq = 0$.

Solution: Let $F = 2zx - px^2 - 2qxy + pq = 0$

The Charpit's auxillary equations are:

$$\frac{dx}{\frac{-\partial F}{\partial p}} = \frac{dy}{\frac{-\partial F}{\partial q}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{df}{0}$$

Here, $\frac{dp}{2z-2ay} = \frac{dq}{0} = \frac{dz}{px^2-pq+2xyq-pq} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dF}{0}$

$\therefore dq = 0 => q = a$

Substituting $q = a$ in the given equation, we have

$2zx - px^2 - 2axy + pa = 0$

$p(x^2 - a) = 2x(z - ay)$

$=> p = \frac{2x(z-ay)}{x^2-a}$

Substituting these values of p, q in dz, we have

$dz = \frac{2x(z-ay)}{x^2-a}dx + ady$

$\frac{dz-ady}{z-ay} = \frac{2xdx}{x^2-a}$

Integrating, we get

$\log(z - ay) = \log(x^2 - a) + \log b$

$=> z - ay = b(x^2 - a)$

$=> z = ay + b(x^2 - a)$ which is the complete integral of the given equation.

8.5 Summary

Understanding and solving second-order PDEs is crucial for modeling and analyzing complex physical systems, predicting their behavior, and designing optimal engineering solutions. These equations play a fundamental role in diverse areas of science and technology, contributing to advancements in fields ranging from aerospace engineering to medical imaging.

8.6 Keywords

- Partial differential Equations
- Order of a Partial differential Equations
- Lagrange's Method
- Charpit's Method

8.7 Self-Assessment Questions

1. How do boundary value problems (BVPs) and initial value problems (IVPs) arise in the context of second-order PDEs, and what techniques are used to solve them?
2. What role do boundary conditions and initial conditions play in determining solutions to second-order PDEs?
3. Can you explain the concept of characteristic curves or surfaces in the context of hyperbolic and parabolic second-order PDEs?
4. How do numerical methods, such as finite difference, finite element, and spectral methods, contribute to the solution of second-order PDEs?

8.8 Case Study

1. Which of the following represents a linear partial differential equation?

A) $u_{xx} + u_{yy} = u$

B) $u_{xx} - u_{yy} = \sin(xy)$

C) $u_x + u_y = u^2$

D) $u_x u_y = u$

2. The heat equation, describing the flow of heat in a given region over time, is an example of which type of partial differential equation?

A) Elliptic

B) Parabolic

C) Hyperbolic

D) Bilinear

3. Which of the following methods is commonly used to solve homogeneous linear partial differential equations with constant coefficients?

A) Method of characteristics

B) Fourier transform

C) Separation of variables

D) Laplace transform

4. Consider the partial differential equation$u_{tt} - c^2u_{xx} = 0$, where c is a constant. What type of equation is this?

A) Parabolic

B) Hyperbolic

C) Elliptic

D) Transcendental

8.9 References

1. Evans, L. C. (2020). Partial Differential Equations (2nd ed.). American Mathematical Society.
2. Strauss, W. A. (2018). Partial Differential Equations: An Introduction. John Wiley & Sons.

Unit-9

Classification of Second Order PDEs

Learning Objectives:

- Understand the concept of heat diffusion and its mathematical representation.
- Grasp the concept of wave propagation and its mathematical description.
- Understand the concept of harmonic functions and their relation to the Laplace equation.

Structure:

9.1 Wave Equations

Essentially, the one-dimensional wave equation derives from the simplest simple case of motion of a stretched string, or more specifically, its transverse vibrations, like those generated by the string of a musical instrument. Assume that a string with x = 0 and x = L is positioned down the x-axis, stretched, and then fastened at both ends. The string is then released, deflected, and given time (t = 0) to vibrate. The string's deflection, u, is the quantity of interest at any position x, $0 < x \leq L$, and at any time t>0. u = u(x, t) is written. The graphic shows the string's potential displacement at a given time t.

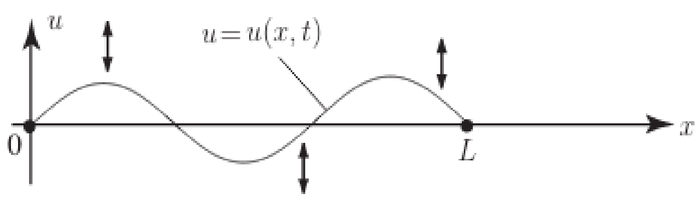

Figure 9.1 : Possible displacement of string

Subject to various assumptions :

1. ignoring air resistance and other dampening factors
2. Ignore the string's weight
3. at any given time, the string's tension is tangential to its curvature.
4. that there are little transverse oscillations produced by the string, meaning that each particle travels just vertically and has a modest deflection and slope at each place.

it can be shown, by applying Newton's Law of motion to a small segment of the string, that u satisfies the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

where

$$c^2 = \frac{T}{\rho}, \ \rho$$

being the mass per unit length of the string and T being the (constant) P horizontal component of the tension in the string.

9.2 Important PDEs in science and engineering:-

1. Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

where f(x, y) is a given function.

2. Helmholtz's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

which arises in wave theory.

3. Schrodinger's equation

$$-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) = E\psi$$

h is Planck's constant

4. Transverse vibrations in a homogeneous rod

$$a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

where u(x, t) is the displacement at time t of the cross-section through x.

Except from the final example, which is fourth order, all of the PDEs we have studied are second order since those are the highest order derivatives that can emerge.

9.3 Summary

These equations, which provide light on processes including heat transfer, wave propagation, and steady-state distributions of scalar fields, are essential to physics and engineering. Their derivations entail the application of fundamental physics concepts to particular physical scenarios, such as conservation laws and Newton's laws of motion.

9.4 Keywords

- Heat diffusion
- Thermal diffusivity

- Infinitesimal element
- Wave speed
- Divergence
- Gradient

9.5 Self Assessment questions

Q1. The heat equation is derived from:

a) Newton's second law

b) Fourier's law of heat conduction and the conservation of energy

c) Maxwell's equations

d) Hooke's law

Q2. The wave equation describes the propagation of waves through a medium by utilizing:

a) Ohm's law

b) Newton's second law

c) Boyle's law

d) Coulomb's law

Q3. The Laplace equation describes:

a) The distribution of temperature over time in a material

b) The propagation of waves through a medium

c) Steady-state phenomena with no sources or sinks of a scalar field

d) The behavior of magnetic fields

Q4. The thermal diffusivity (α) in the heat equation represents:

a) The rate of change of temperature with respect to time

b) The rate of heat transfer

c) The thermal conductivity of the material

d) The rate of change of temperature with respect to position

Q5. In the wave equation, the wave speed (ccc) is determined by:

a) The density of the medium

b) The tension in the medium

c) Both the density of the medium and the tension in the medium

d) Neither the density nor the tension in the medium

9.6 Case Study

Consider a rod of length L with its ends kept at zero temperature. The initial temperature distribution along the rod is given by u(x,0)=f(x)

1. Derive the heat equation for the temperature distribution u(x,t)) in the rod.
2. Explain the physical meaning of each term in the heat equation.

9.7 References

1. "Elementary Differential Equations and Boundary Value Problems" by William E. Boyce and Richard C. DiPrima
2. "Advanced Engineering Mathematics" by Erwin Kreyszig

Unit-10

The Cauchy Problem

Learning Objectives:

- The primary learning objective of Cauchy-Euler equations is to understand how to solve second-order linear ordinary differential equations with variable coefficients.
- Understand the characteristic form of Cauchy-Euler equations, which involves terms with derivatives of different orders multiplied by powers of the independent variable.

Structure:

10.1 Cauchy-Euler Equations and Special Cases

The differential equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_0 y = 0$$

is called the Cauchy-Euler differential equation of order n. The symbols a_i, i=0,...,n are constants and $a_n \neq 0$.

The second-order Cauchy-Euler formula

$$ax^2 y'' + bxy' + cy = 0$$

A theoretical justification for examining the Cauchy-Euler equation is its uniqueness as a differential equation with non-constant coefficients and a known closed-form solution. The change in variables (x, y) → (t, z) provided by equations is the cause of this fact.

$$x = e^t, \quad z(t) = y(x),$$

The Cauchy-Euler equation is transformed into a constant-coefficient differential equation. Like the constant-coefficient equations, the Cauchy-Euler equations also have closed-form solutions.

Example 1: Solve

$$2x^2 y'' + 4xy' + 3y = 0,$$

verifying general solution

$$y(x) = c_1 x^{-1/2} \cos\left(\frac{\sqrt{5}}{2} \ln|x|\right) + c_2 e^{-t/2} \sin\left(\frac{\sqrt{5}}{2} \ln|x|\right).$$

Solution: The characteristic equation $2r(r-1) + 4r + 3 = 0$ can be obtained as follows:

$2x^2 y'' + 4xy' + 3y = 0$	Given differential equation.
$2x^2 r(r-1)x^{r-2} + 4xrx^{r-1} + 3x^r = 0$	Use **Euler's substitution** $y = x^r$.
$2r(r-1) + 4r + 3 = 0$	Cancel x^r. **Characteristic equation** found.
$2r^2 + 2r + 3 = 0$	Standard quadratic equation.
$r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} i$	Quadratic formula complex roots.

7.5 Cauchy-Euler Substitution:-

The second step is to use y(x) = z(t) and

$x = e^t$ to transform the differential equation. By Theorem 5,

$$2(d/dt)^2 z + 2(d/dt)z + 3z = 0,$$

a constant-coefficient equation. Because the roots of the characteristic equation $2r^2 + 2r + 3 = 0$ are $r = -1/2 \pm \sqrt{5}i/2$, then the Euler solution atoms are

$$e^{-t/2}\cos\left(\frac{\sqrt{5}}{2}t\right), \quad e^{-t/2}\sin\left(\frac{\sqrt{5}}{2}t\right).$$

Back-substitute $x = e^t$ and $t = \ln|x|$ in this equation to obtain two independent solutions of $2x^2y'' + 4xy' + 3y = 0$:

$$x^{-1/2}\cos\left(\frac{\sqrt{5}}{2}\ln|x|\right), \quad e^{-t/2}\sin\left(\frac{\sqrt{5}}{2}\ln|x|\right).$$

Substitution Details. Because $x = e^t$, the factor $e^{-t/2}$ is written as $(e^t)^{-1/2} = x^{-1/2}$. Because $t = \ln|x|$, the trigonometric factors are back-substituted like this: $\cos\left(\frac{\sqrt{5}}{2}t\right) = \cos\left(\frac{\sqrt{5}}{2}\ln|x|\right)$.

10.3 Summary

Comprehending these particular scenarios is essential for resolving Cauchy-Euler equations and using them for diverse practical issues in domains like physics, engineering, and economics.

10.4 Keywords

- Cauchy-Euler Equations
- Second-Order Linear ODEs
- Distinct Real Roots
- Repeated Real Roots
- Complex Roots

10.5 Self Assessment Questions

1. What is a Cauchy-Euler equation?
2. Write the general form of a second-order Cauchy-Euler differential equation.

3. How does the Cauchy-Euler equation differ from a standard linear differential equation?
4. What substitution is typically used to solve a Cauchy-Euler equation?
5. Explain the purpose of the substitution $x=e^t$ in solving a Cauchy-Euler equation.
6. How do you solve a Cauchy-Euler equation with distinct real roots?

10.6 Case Study

1. Describe a physical or engineering system where Cauchy-Euler equations naturally arise.
2. Identify the main characteristics of a problem that suggests the use of a Cauchy-Euler equation.

10.7 References

1. Wiggins, S). Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer.
2. Hale, J. K., &Koçak, H. Dynamics and Bifurcations. Springer.

Unit-11

Initial Boundary Value Problems

Learning Objectives:

- Understand the stability, consistency, and convergence criteria for numerical solutions of IBVPs.
- Understand the impact of boundary and initial conditions on the behavior and solution of differential equations.
- Gain exposure to advanced topics related to IBVPs, such as non-linear IBVPs, multi-dimensional IBVPs, and stochastic IBVPs.

Structure:

11.1 Solving the wave equation for the infinite string:-

Initial Boundary Value Problems (IBVPs) in differential equations are mathematical problems that involve finding a function that satisfies a differential equation within a given domain and also meets specified initial conditions and boundary conditions. These problems are crucial in the study of physical systems described by partial differential equations (PDEs) and ordinary differential equations (ODEs).

Here's an overview of IBVPs:

1. Initial Conditions

Initial conditions specify the state of the system at the beginning of the observation period. For example:

- In ODEs, an initial condition might specify the value of the function and possibly its derivatives at a specific point.

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \text{etc.}$$

- In PDEs, initial conditions might specify the value of the function over a spatial domain at an initial time.

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

2. Boundary Conditions

Boundary conditions specify the behavior of the function on the boundary of the spatial domain. Common types include:

- **Dirichlet Boundary Conditions:** The value of the function is specified on the boundary.

$$u(a,t) = \alpha, \quad u(b,t) = \beta$$

- **Neumann Boundary Conditions:** The derivative of the function is specified on the boundary.

$$\frac{\partial u}{\partial x}(a,t) = \gamma, \quad \frac{\partial u}{\partial x}(b,t) = \delta$$

- **Mixed Boundary Conditions:** A combination of Dirichlet and Neumann conditions.

$$u(a,t) = \alpha, \quad \frac{\partial u}{\partial x}(b,t) = \delta$$

3. Examples of IBVPs

Example 1:

A popular PDE for simulating the temperature or heat distribution over time in a given area is the heat equation. It may be written as follows for a one-dimensional rod:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with initial condition:

$$u(x,0) = f(x)$$

and boundary conditions:

$$u(0,t) = u(L,t) = 0$$

Example 2: Wave Equation

The wave equation simulates how waves, like light or sound waves, travel through a medium.:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

with initial conditions:

$$u(x,0) = g(x), \quad \frac{\partial u}{\partial t}(x,0) = h(x)$$

and boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0$$

4. Methods of Solution

Separation of Variables

According to this approach, the answer can be expressed as the product of functions that are all dependent on the same coordinate. For example, for the heat equation:

$$u(x,t) = X(x)T(t)$$

By substituting into the PDE and separating the variables, we obtain two ordinary differential equations that can be solved independently.

11.2 Semi-Infinite String with a fixed end:

The problem of a semi-infinite string with a fixed end is a classic example in the study of wave equations in mathematical physics. It involves finding the displacement of a string that extends infinitely in one direction and is fixed at one end. This scenario is governed by the wave equation.

Problem Setup

Consider a string that is fixed at x=0 and extends infinitely in the positive x-direction. The wave equation governing the motion of the string is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary and Initial Conditions

1. Boundary Condition at the Fixed End: The displacement is zero at the fixed end (x=0)
u(0,t) = 0 for all t > 0

2. Initial Conditions: These specify the initial displacement and velocity of the string:
u(x, 0) = f(x) for x>0

$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for } x \geq 0$$

Reflection Method and D'Alembert's Solution

1. Extension by Reflection: Define the displacement function for < 0 such that the extended function is odd:

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{for } x \geq 0 \\ -u(-x,t) & \text{for } x < 0 \end{cases}$$

2. Wave Equation Solution: For the extended function u(x, t), the wave equation remains thesame, but now it is defined on the entire real line $-\infty < x < \infty$.

3. D'Alembert's Solution: The general solution of the wave equation on an infinite domain is given by:

$$\tilde{u}(x,t) = F(x-ct) + G(x+ct)$$

where F and G are determined by the initial conditions.

4. Initial Conditions: To find F and G:

$$\tilde{u}(x,0) = F(x) + G(x) = \begin{cases} f(x) & \text{for } x \geq 0 \\ -f(-x) & \text{for } x < 0 \end{cases}$$

$$\frac{\partial \tilde{u}}{\partial t}(x,0) = -cF'(x) + cG'(x) = \begin{cases} g(x) & \text{for } x \geq 0 \\ -g(-x) & \text{for } x < 0 \end{cases}$$

5. Solving for F and G:

$$F(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\int_0^x g(s)\,ds\right)$$

$$G(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\int_0^x g(s)\,ds\right)$$

6. Complete Solution:

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(s)\,ds\right)$$

This approach allows for solving the wave equation with the given initial and boundary conditions for a semi-infinite string with a fixed end.

11.3 Semi-Infinite String with a Free end:

The problem of a semi-infinite string with a free end involves solving the wave equation for a string that is fixed at one end and extends infinitely in one direction, with the other end being free. This type of problem is governed by the same wave equation but with different boundary conditions reflecting the free end's behavior.

Consider a string fixed at x=0and extending infinitely in the positive x-direction. The wave equation governing the motion of the string is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where u(x, t) represents the displacement of the string at position & and time t, and c is the wave speed.

Boundary and Initial Conditions

1. Boundary Condition at the Fixed End: The displacement is zero at the fixed end (x = 0):

$u(0,t) = 0$ for all $t > 0$

2. Boundary Condition at the Free End: The spatial derivative of the displacement is zero at the free end ($x \to \infty$):

3. Initial Conditions: These specify the initial displacement and velocity of the string:

$u(x, 0) = f(x)$ for $x>0$

$$\frac{\partial u}{\partial x} \to 0 \quad \text{as } x \to \infty$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \quad \text{for } x \geq 0$$

Reflection Method and D'Alembert's Solution

1. Extension by Reflection: Similar to the case of a fixed end, we can extend the problem to the entire real line by reflecting the function about the fixed end. However, since the other end is free, we need to ensure that the reflected wave respects the free-end boundary condition.

2. Defining the Extended Function:

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{for } x \geq 0 \\ -u(-x,t) & \text{for } x < 0 \end{cases}$$

3. Ensuring the Free-End Condition: We need to ensure that ũ satisfies the wave equation for all x Є R. The boundary condition at x = 0 is automatically satisfied because of the reflection. The condition at the free end (as → ∞) translates into ensuring that the spatial derivative of the solution does not become unbounded.

11.4 Summary

The solution for the semi-infinite string with a free end is constructed by extending the problem to an infinite domain, ensuring the boundary conditions are satisfied by reflecting the function, and applying D'Alembert's solution. This approach effectively handles the initial conditions and the boundary condition at the fixed end, providing the displacement of the string over time.

11.5 Keywords

- Initial Conditions
- Boundary Conditions
- Differential Equations
- Ordinary Differential Equations
- Partial Differential Equations

11.6 Self-Assessment Questions

1. What is an Initial Boundary Value Problem (IBVP)?
2. What distinguishes an IBVP from a Boundary Value Problem (BVP)?
3. What are the typical components of an IBVP?
4. Can you give an example of a physical phenomenon modeled by an IBVP?
5. How do initial conditions differ from boundary conditions in an IBVP?
6. What role does the domain of the problem play in an IBVP?
7. What is a common method for solving IBVPs numerically?
8. How does the method of separation of variables apply to IBVPs?

11.7 Case Study

A manufacturing plant is monitoring the temperature distribution along a metal rod that is being heated at one end while the other end is kept at a constant lower temperature. The rod is 1 meter long. The heat conduction in the rod is modeled by the heat equation, a partial differential equation, which must be solved to understand how the temperature evolves over time along the length of the rod.

Problem Statement:The temperature distribution u(x, t) along the rod is governed by the heat equation: $u_t = \alpha u_{xx}$

whereα is the thermal diffusivity of the rod's material. The rod has the following initial and boundary conditions:

- Initial temperature distribution: u(x, 0) = f(x)
- Boundary condition at the heated end (x =0): u(0,t) = T_0
- Boundary condition at the cooler end (x = = 1): u(1,t) = T_1

11.8 References

1 Wiggins, S. Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer.

2 Hale, J. K., &Koçak, H. Dynamics and Bifurcations. Springer.

Unit-12

Non-Homogeneous Wave Equation

Learning Objectives:

- Differentiate between homogeneous and non-homogeneous wave equations.
- Understand the role of initial and boundary conditions in solving the non-homogeneous wave equation.

Structure:

12.1 Non-Homogeneous Wave Equation:

Now we consider the nonhomogeneous (NH) wave equation on the real line

$$u''_{tt} - c^2 u''_{xx} = f(x,t)$$

subject to the following initial conditions (IC): u(x, 0) = g(x), u'_t(x, 0) = (x).

Remark: Solution of the NH equation can be represented as a sum of two other solutions:

Problem I: the non -homogeneous wave equation V_{tt} - c2v_{xx} = f with homogeneous IC:

v(x, 0) = 0,v{(x, 0) = 0,

Problem II: the homogeneous wave equation utt - c2uxx = 0 with nonhomogeneous IC:

u(x, 0) = g(x),u_t(x, 0) = h(x).

Thus, it suffices only to consider the first problem. We apply the method due to Duhamel (Jean Marie Constant Duhamel (1797-1872), a French mathematician).

Namely, consider an auxiliary problem

$$U''_{tt} - c^2 U''_{xx} = 0, \qquad \text{or} \quad x \in \mathbb{R}, \quad t > 0$$

$$U(x,0,s) = 0, \qquad U'_t(x,0,s) = f(x,s), \qquad \text{for} \quad x \in \mathbb{R}, \ s > 0.$$

Here f(x, s) is the right hand side in our equation given above.

Duhamel's principle

Assume that U(x, t, s) is a C^2-function of x Є R and t> 0, continuous in s, s>0. If U solves the

$$u(x,t) = \int_0^t U(x, t-s, s)\,ds.$$

Proof. Apply differentiation with respect to parameter,

$$\frac{\partial}{\partial t}\int_{a(t)}^{b(t)} F(x,t)dt = F(b(t),t)b'(t) - F(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}F(x,t)dt.$$

This formula holds if F and F'_t are continuous. In our case $F(x,t) = U(x, t-s, s)$, thus

$$u_t(x,t) = U(x, t-t, t)\cdot 1 - U(x, t-0, 0)\cdot 0 + \int_0^t \frac{\partial}{\partial t}U(x, t-s, s)ds,$$

and applying $U(x,0,s) = 0$, we find

$$u_t(x,t) = \int_0^t U_t(x, t-s, s)ds.$$

above auxiliary problem, then solution of the Problem I is given by

Differentiate again and apply the second initial condition:

$$u_{tt}(x,t) = U_t(x,0,t) + \int_0^t U_{tt}(x,t-s,s)ds = f(x,t) + \int_0^t U_{tt}(x,t-s,s)ds.$$

Differentiation with respect to x yields

$$u_{xx}(x,t) = \int_0^t U_{xx}(x,t-s,s)ds,$$

hence, combining the found formulas we get

$$u_{tt} - c^2 u_{xx} = f(x,t) + \int_0^t (U_{tt}(x,t-s,s) - c^2 U_{xx}(x,t-s,s))ds = f(x,t).$$

12.2 Method of separation of variables:-

With specified initial and boundary conditions, the method of separation of variables is a potent tool for solving linear partial differential equations (PDEs) of order two. Here is a detailed explanation of this method:

1. **Assumption**: Assuming that the given PDE's solution can be expressed as a product of functions, each of which depends only on one variable, is the first step in the separation of variables technique. For instance, we suppose the following solution for a PDE with two variables, x and t:

   u(x,t) = X(x)T(t)

   where X(x) is a function of x alone, and T(t) is a function of t alone.

   2. Substitution: Substitute the assumed solution into the original PDE. This substitution will usually result in an equation where the terms involving a are separated from the terms involving t.

   3. Separation: Once substitution is complete, the variables are split apart so that the functions on either side of the equation are solely dependent on an or t. Any value of t and. will result in these two sides being equal, hence each side must equal a constant. Two ordinary differential equations (ODEs) are produced as a result of this procedure.

4. Solve the ODES: Solve these ODEs separately. The solutions to these ODEs will involve arbitrary constants that can be determined using initial and boundary conditions.

Example 1:

Let's illustrate this with a simple example: the heat equation $u_t = ku_{xx}$

1. Assume: Assume u(x, t) = X(x)T(t).
2. Substitute: Substitute into the heat equation:

$$\frac{\partial}{\partial t}(X(x)T(t)) = k\frac{\partial^2}{\partial x^2}(X(x)T(t))$$

$$X(x)\frac{dT(t)}{dt} = kT(t)\frac{d^2X(x)}{dx^2}$$

12.3 Solving the Heat Conduction problem:-

The heat conduction problem, also known as the heat equation problem, involves finding the temperature distribution in a given domain over time.

Boundary Conditions

Assume the rod is of length L and the ends are maintained at a fixed temperature (usually taken as zero for simplicity):

u(0,t) = 0 and u(L,t) = 0 for all t

Initial Condition

The initial temperature distribution along the rod is given by:

u(x, 0) = f(x)

Solution Using Separation of Variables

1. Assume a Solution: Assume a solution of the form:

u(x,t) = X(x)T(t)

2. Substitute: Substitute this into the heat equation:

$$X(x)\frac{dT(t)}{dt} = kT(t)\frac{d^2X(x)}{dx^2}$$

3. Separate Variables: Divide both sides by kX (x)T(t):

$$\frac{1}{k}\frac{1}{T(t)}\frac{dT(t)}{dt} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\lambda$$

Here, $-\lambda$is a separation constant. This gives us two ordinary differential equations:

$$\frac{dT(t)}{dt} + \lambda kT(t) = 0 \quad \text{and} \quad \frac{d^2X(x)}{dx^2} + \lambda X(x) = 0$$

4. Solve the Spatial ODE: Consider the spatial part:

$$\frac{d^2X(x)}{dx^2} + \lambda X(x) = 0$$

with boundary conditions X(0)=0 and X (L) = 0.

The solutions are:

$$X(x) = A\sin\left(\frac{n\pi x}{L}\right)$$

where $\lambda = \left(\frac{n\pi}{L}\right)^2$ and n is a positive integer.

12.4 Summary:

Scientists and engineers may forecast and analyze complicated wave behaviors in a variety of physical systems by comprehending and solving the non-homogeneous wave equation. This can provide light on how external factors impact wave interactions and propagation.

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12.5Keywords

- Wave Equation
- Non-homogeneous
- Source Term
- Initial Conditions
- Boundary Conditions

12.6 Self-Assessment Questions

1. What is the role of the source term f(x,t)f(x,t)f(x,t) in the non-homogeneous wave equation?
2. Give an example of a physical situation that can be modeled by a non-homogeneous wave equation.
3. What is the method of undetermined coefficients?
4. Explain how the method of separation of variables can be used for solving the non-homogeneous wave equation.
5. Describe D'Alembert's solution to the non-homogeneous wave equation.

12.7 Case Study

An electromagnetic wave propagates along a transmission line, influenced by external sources such as power sources and antennas. The wave behavior is governed by the non-homogeneous wave equation.

Questions:

1. **Equation Formulation:** Define the non-homogeneous wave equation governing the propagation of electromagnetic waves in the transmission line. What factors determine the form of the source term $(x,)$?
2. **Analytical Solution Techniques:** Explore possible analytical methods to solve the non-homogeneous wave equation for the electromagnetic waves. How would you handle the non-homogeneous term in the solution process?

3. **Numerical Simulation:** Propose a numerical method to simulate the propagation of electromagnetic waves in the transmission line. Discuss the implementation of this method and how it accounts for the non-homogeneous source term.

12.8 References

1. Wiggins, S. Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer.
2. Hale, J. K., &Koçak, H. Dynamics and Bifurcations. Springer.